Nonlinear System

Lecture 23

04/23/13

$$\int_0^T y^T(t)\, u(t)\, dt \geq 0$$ (standard notion of passivity)

has to hold for all times T and all input trajectories

accounts for IC's

$$\underbrace{\int_0^T y^T(t)\, u(t)\, dt}_{\langle y_T, u_T\rangle} \geq \begin{cases} -\beta & \text{passive (P)} \\ \delta \langle u_T, u_T\rangle - \beta & \text{input strictly passive (ISP)} \\ \varepsilon \langle y_T, y_T\rangle - \beta & \text{output strictly passive (OSP)} \end{cases}$$

→ State space characterization

$$\dot{V}(x) \leq \begin{cases} y^T u & \cdots \text{(P)} \\ -\delta u^T u + y^T u & \cdots \text{(ISP)} \\ -\varepsilon y^T y + y^T u & \cdots \text{(OSP)} \end{cases}$$

$$\dot{x} = f(x) + g(x) u$$
$$y = h(x)$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \leq h^T(x) . u \qquad (*)$$

$(*) \Leftrightarrow$ $\frac{\partial V}{\partial x} f(x) \leq 0$ (1) → stability in the sense of Lyap.

$\frac{\partial V}{\partial x} g(x) = h^T(x)$ (2) additional condition

For linear systems $\qquad V(x) = \frac{1}{2} x^T P x$

$$A^T P + P A \leq 0 \qquad (1)$$

$$P B = C^T \qquad (2)$$

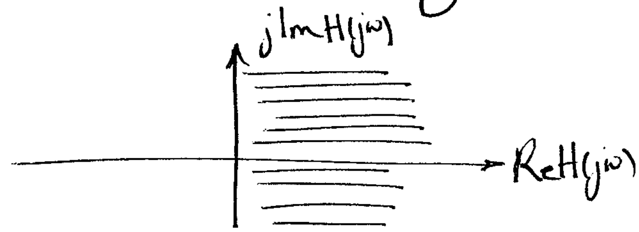
Implications of positive realness :

1) If $H(s)$ is PR $\Rightarrow$ stable

If $H(s)$ is SPR $\Rightarrow$ asymptotically stable

2) $|\measuredangle H(j\omega)| < 90^\circ$ (Bode plot)

Nyquist plot lies in the right half plane



3) Relative degree of H is either 0 or 1

$H(s) = \dfrac{P(s)}{q(s)}$ $\qquad$ order of q - order of p = (# of poles) - (# of zeros)

= relative degree

KYP Lemma (Kalman, Yakubovich, Popov)

(Positive Real Lemma)

let $H(s) = C(sI-A)^{-1}B$ with

$Re(\lambda_i(A)) \leq 0$ and (A,B) controllable

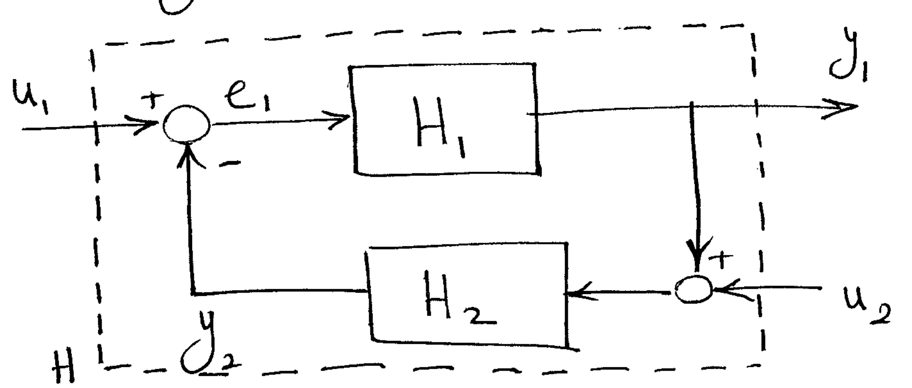
then $H(s)$ is PR iff $\exists P = P^T > 0$ s.t. $A^TP + PA \leq 0$

and $PB = C^T$

$H(s)$ is SPR iff $\exists P = P^T > 0$ s.t. $A^TP + PA < 0$

$PB = C^T$

Passivity thm.



$H: \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \to \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

a) H_i: passive with storage functions V_1 & V_2

H_1 : $\dot{V}_1 \leq e_1^T y_1 = (u_1 - y_2)^T y_1 = u_1^T y_1 - y_2^T y_1$ ... (1)

H_2 : $\dot{V}_2 \leq e_2^T y_2 = (u_2 + y_1)^T y_2 = u_2^T y_2 + y_1^T y_2$ ... (2)

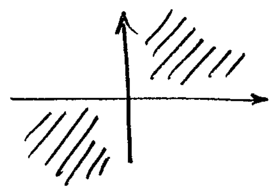
(1)+(2) with $V := V_1 + V_2$

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \leq u_1^T y_1 + u_2^T y_2 = [u_1^T \ u_2^T]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = u^T y$$

This characterization is very general. H(transfer functions) can be anything, even infinit dimensional.

b) H_1: same as in (a) but now let H_2 be memoryless nonlinearity with:

H_2: $y_2^T e_2 \geq 0$



(passivity)

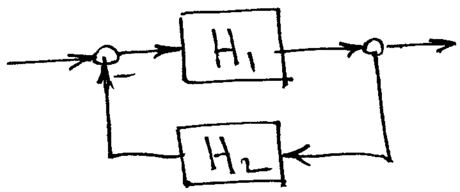
$$y_2^T(y_1 + u_2) \geq 0$$

$$-y_2^T y_1 \leq y_2^T u_2 \qquad (3)$$

(3) $\to$ (1) $\quad \dot{V}_1 \leq u_1^T y_1 + u_2^T y_2 = u^T y$

$\Rightarrow$ feedback interconnection with storage function V_1.

Linear Systems:



Recall: small-gain

$$|H_1(j\omega)|\,|H_2(j\omega)| < 1 \quad \text{for every } \omega$$

passivity? $\longrightarrow$ info about phase characterizations

H_i: positive real $\quad |\measuredangle H_i(j\omega)| < 90^\circ$

$$\Downarrow$$

$|\measuredangle H_1(j\omega) H_2(j\omega)| < 180^\circ \Rightarrow$ Nyquist plot doesn't cross the real line, i.e. doesn't encircle -1.

Side note : inverse optimality (Kalman '62-63)

LQR:

$K = R^{-1}B^TP$

let $R = I$ (for simplicity)

$K_{LQR} = B^TP \,/^T \longrightarrow PB = K_{LQR}^T$

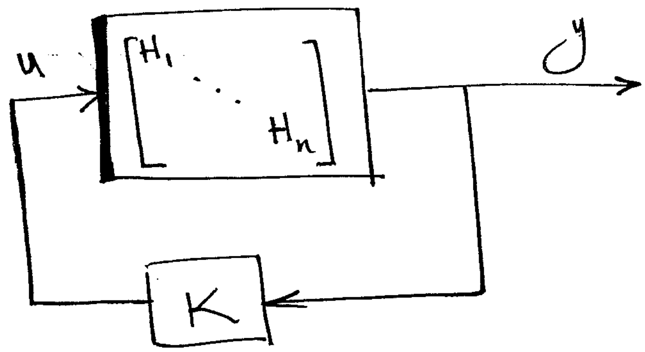
ARE: $$A^TP + PA + Q - \overbrace{PB\,\overset{RR^{-1}}{R^{-1}}}^{K^T}\underbrace{B^TP}_{K} = 0$$

Kalman showed that a given feedback gain K is inversely optimal (Q & R can be recovered) if passivity holds when output matrix C is chosen to be K.

$$(A-BK)^TP + P(A-BK) = -Q - K^TRK$$ (observability gramian)

RE for open loop equation is equivalent to the lyapunov equation for the closed loop system.

A generalization to (potentially) large-scale interconnections:



$$H_i : \quad \dot{x}_i = f(x_i) + g(x_i)\,u_i$$
$$y_i = h(x_i)$$

H_i : SISO

Each H_i : output strictly passive

$$\dot{V}_i \leq -\varepsilon_i y_i^2 + y_i u_i$$

$$u := \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad y := \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \quad ; \; u = Ky$$

let $\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \quad ; \; d_i \geq 0$

$$\leq \sum_{i=1}^{n} d_i(-\varepsilon_i y_i^2 + y_i u_i)$$

We want to derive sufficient conditions for K for the stability of the interconnection

$$D_d := \mathrm{diag}\{d_i\} = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}$$

$$D_q := \mathrm{diag}\{\varepsilon_i\} = \begin{bmatrix} \varepsilon_1 & & \\ & \ddots & \\ & & \varepsilon_n \end{bmatrix}$$

$$\dot{V} \leq y^T(-D_d \cdot D_\varepsilon)y + y^T D_d K y = y^T(-D_d D_\varepsilon + D_d K)y$$

$$= y^T D_d(-D_\varepsilon + K)y$$

$$= \tfrac{1}{2} y^T \left\{(-D_\varepsilon + K)^T D_d + D_d(-D_\varepsilon + K)\right\} y$$

sufficient conditions for stability $\Rightarrow$ existence of diagonal matrix D_d which is positive definite as the sol'n to

$$(-D_\varepsilon + K)^T D_d + D_d(-D_\varepsilon + K) < 0$$

$$\Downarrow$$

stability