Nonlinear Systems

Lecture 22 04/18/13

Last time:

- Hamilton-Jacobi $\neq$
- Bounded Real Lemma (linear systems)
- Small gain thm

Important!

Today:

- Passivity

$H : L_{2e} \to L_{2e}$

H: passive if for any $u \in L_{2e}$ and any $T > 0$

$\langle y_T, u_T \rangle \geq -\beta$ → accounts for initial conditions

$(x(0) \equiv 0 \Rightarrow \langle y_T, u_T \rangle \geq 0)$

$$\langle y_T, u_T \rangle := \langle y, u \rangle_{L_2[0,T]} = \int_0^T y^T(t) u(t) dt$$

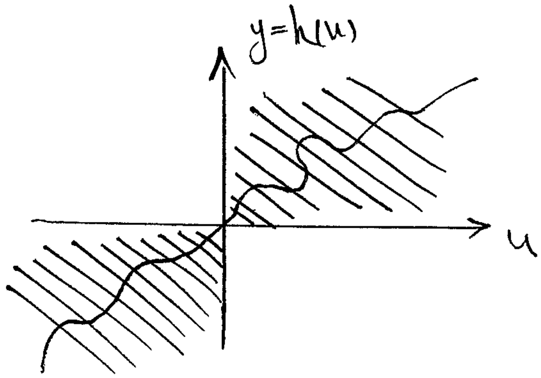
$$\langle y_T, u_T \rangle \geq \begin{cases} -\beta & ; \text{ passive} \\ \delta \langle u_T, u_T \rangle - \beta & ; \text{ input strictly passive} \\ \epsilon \underbrace{\langle y_T, y_T \rangle}_{\|y_T\|_2^2} - \beta & ; \text{ output strictly passive} \end{cases} \qquad (*)$$

Memoryless (static) nonlinearity

$$u \longrightarrow \boxed{h(\cdot)} \longrightarrow y = h(u)$$

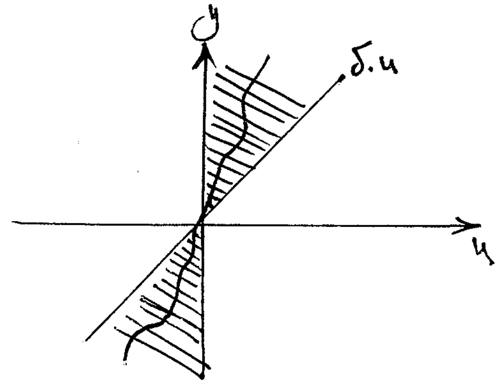
Passive $\quad y \cdot u \geq 0$

$$h(u) \cdot u \geq 0$$



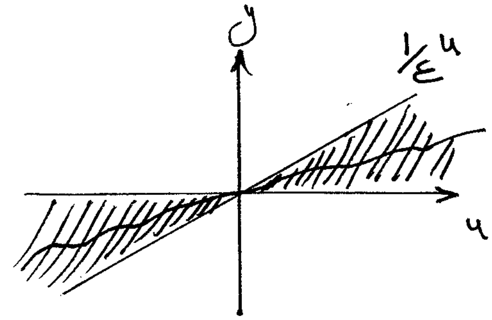
Input strict passivity

$$y \cdot u \geq \delta u^2$$



Output strict passivity

$y \cdot u \geq \varepsilon y^2$



remember that Lp stability had a bow-tie region

State space conditions for passivity:

$$\dot{x} = f(x) + g(x)u$$
$$y = h(x)$$

If there is a continuously differentiable possitive definite storage function $(V(x))$ st.

$$(P) \rightarrow \quad \dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \leq \begin{cases} y^T u & ; \text{ passive} \\ y^T u - \delta u^T u & ; \text{ input strictly passive} \\ y^T u - \varepsilon y^T y & ; \text{ output strictly passive} \end{cases}$$

Proof integrate (P)

$$\underbrace{-V(x(0))}_{\beta} \leq V(x(T)) - V(x(0)) \leq \langle y_T, u_T \rangle - \begin{cases} 0 \\ \delta\langle u_T, u_T \rangle \\ \varepsilon \langle y_T, y_T \rangle \end{cases}$$

move to recover (*)

■

Ex 1.

$$\dot{x} = u$$
$$y = x$$

$$V(x) = \frac{1}{2} x^2$$

$$\boxed{\dot{V}(x)} = x\dot{x} = xu = \boxed{yu} \longrightarrow \text{therefore passive system}$$

Note: if y is velocity and u is forcing.

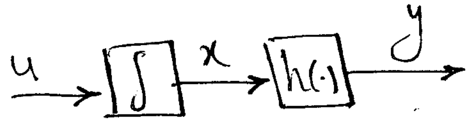
$\Rightarrow$ $y \cdot u$ = velocity $\times$ force = power supplied to the system

there is no internal mechanism for storing energy

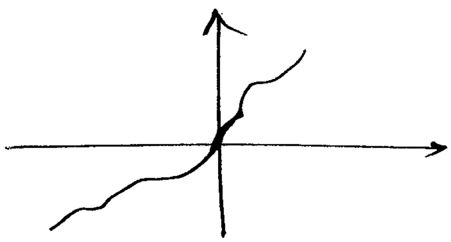
$\Rightarrow$ $V = \frac{1}{2} x^2$ is interpreted as the kinetic energy of the system and the rate of change of kinetic energy is equal to the energy supplied to the system.

Ex 2

$$\dot{x} = u$$
$$y = h(x)$$



$x \cdot h(x) > 0$ for all $x \neq 0$



$$V(x) = \int_0^x h(\xi)\, d\xi \nearrow$$ can be interpreted as potential energy

$$\dot{V}(x) = h(x) \cdot \dot{x} = h(x) \cdot u$$
$$= y \cdot u$$

Ex 3

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{dP(x_1)}{dx_1} - K x_2 + u \end{cases}$$

potential energy

force

velocity

$$y = x_2$$

$$V(x) = P(x_1) + \frac{1}{2} x_2^2$$

(potential) + kinetic

$$\dot{V}(x) = \frac{dP}{dx_1} \dot{x}_1 + x_2 \dot{x}_2 = \cancel{\frac{dP}{dx_1}} x_2 + x_2 \left\{ -\cancel{\frac{dP}{dx_1}} - K x_2 + u \right\}$$

$$= -K x_2^2 + x_2 u$$

$$= -K y^2 + y u \leq y \cdot u$$

output strictly passive

Consequences of passivity

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g(x) u \leq y^T u \qquad (*)$$

1. $\frac{\partial V}{\partial x} f(x) \leq 0$ (set $u=0$ in $(*)$)

$\downarrow$

stability in the sence of Lyapunov

2. $\frac{\partial V}{\partial x} g(x) = y^T$ (otherwise, we can choose u to make)

$$\dot{V} = \frac{\partial V}{\partial x} f(x) + \left(\frac{\partial V}{\partial x} g(x) - y^T\right) u > 0$$

Note! If $(*)$ holds $\Longleftrightarrow$ (1) and (2) hold.

linear case:

$$V(x) = \tfrac{1}{2} x^T P x$$

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

(1) $\frac{\partial V}{\partial x} f(x) \leq 0 \Longleftrightarrow \frac{1}{2} x^T (A^T P + PA) x \leq 0$

$A^T P + PA \leq 0$ standard stability

(2) $\frac{\partial V}{\partial x} g(x) = x^T P B = x^T C^T$

$\boxed{PB = C^T}$ restriction needed to get passivity in addition to stability

Passivity : stability $\rightarrow$ $A^TP + PA \leq 0$

$\oplus$

input-output constraint $\rightarrow$ $PB = C^T$

B: input matrix; C: output matrix

input-output constraint $\rightarrow$ structural constraint

$$\boxed{\text{For LTI systems, passivity} \Updownarrow \text{positive realness}}$$

Def : A proper rational transfer function $H(s)$ with real coefficients is positive real (PR) if :

proper: $\deg\{q(s)\} \geq \deg\{p(s)\}$

rational: ratio of two polynomials $\frac{p(s)}{q(s)}$

a) poles of $H(s)$ satisfy $Re(\lambda_i) \leq 0$.

poles on $j\omega$-axis are simple (not repeated) and the associated residues (coefficients of partial fractional expansion) are non-negative.

b) $\mathrm{Re}\{H(j\omega)\} \geq 0 \quad , \forall \omega \in \mathbb{R}$

It is strictly positive real (SPR) if

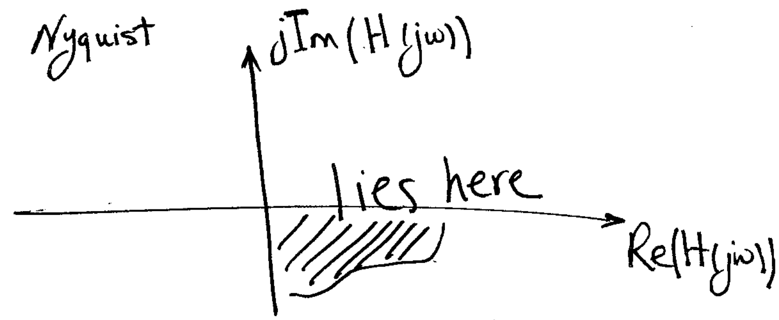
$H(s-\varepsilon)$ is PR for some $\varepsilon > 0$.

Ex

a) $\frac{1}{s}$: PR but not SPR

b) $-\frac{1}{s}$: Not PR , because of negative residue

c) $\frac{1}{s+a}$, $a>0$



$$H(j\omega) = \frac{1}{a+j\omega} = \frac{a-j\omega}{a-j\omega}\,\frac{1}{a+j\omega} = \underbrace{\frac{a}{a^2+\omega^2}}_{\geq 0} - \frac{j\omega}{a^2+\omega^2}$$

$\Rightarrow$ SPR