Nonlinear Systems

Lecture 21 04/16/13

Last time

Input output stability

Finite gain $\mathcal{L}_p$ stability

$$\|y_T\|_p \leq K\|u_T\|_p + \beta$$

$\forall u \in \mathcal{L}_{pe}$ $\forall T \geq 0$

$$\left(y_T = \begin{cases} y(t) & 0 \leq t \leq T \\ 0 & \text{o.w.} \end{cases} \right.$$

A state-space condition for $\mathcal{L}_2$ stability of

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \qquad (*) \\ y &= h(x) \end{aligned}$$

If there is a cts diff'ble, pos definite $V(x)$

st.

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0 \quad (HJ)$$

then ⊛ has an $\mathcal{L}_2$ gain $\leq \gamma$

→ Hamilton-Jacobi inequality

Proof: let (HJ) hold

$$\dot{V}(x) = \frac{\partial V}{\partial x} \dot{x} = \underbrace{\frac{\partial V}{\partial x} f(x)}_{L_f V} + \boxed{\underbrace{\frac{\partial V}{\partial x} g(x)}_{L_g V} u}$$

$$\dot{V} \leq \frac{\partial V}{\partial x} f(x) + \frac{1}{2\alpha} \frac{\partial V}{\partial x} g(x) g^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{\alpha}{2} u^T u$$

$$\left[\; a.b \leq \frac{1}{2\alpha} a^2 + \frac{\alpha}{2} b^2 \iff 0 \leq \left(\frac{a}{\sqrt{\alpha}} - \sqrt{\alpha}\, b\right)^2 \; \right]$$

Now from (HJ) with $\alpha = \gamma^2$

$$\Rightarrow \quad \dot{V} \leq -\frac{1}{2} h^T(x) h(x) + \frac{\gamma^2}{2} u^T u$$

$$= -\frac{1}{2} y^T(t) y(t) + \frac{\gamma^2}{2} u^T(t) u(t)$$

Now, integrate from 0 to T :

$$V(x(T)) - V(x(0)) \leq -\frac{1}{2}\|y_T\|_2^2 + \frac{\gamma^2}{2}\|u_T\|_2^2$$

$$\Downarrow$$

$$-V(x(0)) \leq V(x(T)) - V(x(0))$$

$$\Downarrow$$

$$\|y_T\|_2^2 \leq \gamma^2 \|u_T\|_2^2 + 2V(x(0))$$

we have: $\sqrt{a^2+b^2} \leq |a| + |b|$

$$\Rightarrow \quad \|y_T\|_2 \leq \underbrace{\gamma}_{K} \|u_T\|_2 + \underbrace{\sqrt{2V(x(0))}}_{\beta}$$

Note!
Lyapunov like functions that are used to establish input-output stability are known as "storage functions".

For linear systems

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

(HJ) holds with

$$V(x) = \frac{1}{2} x^T P x$$

and it simplifies to

$$x^T \left(A^T P + PA + \frac{1}{\gamma^2} PBB^T P + C^T C \right) x \leq 0$$

$$\Downarrow$$

$$A^T P + PA + \frac{1}{\gamma^2} PBB^T P + C^T C < 0$$

negative definite

* Bounded Real Lemma :

Suppose A is Hurwitz ($\lambda_i(A) < 0 \quad i=1,\dots,n$)

and let γ^* denote the L_2-induced gain of :

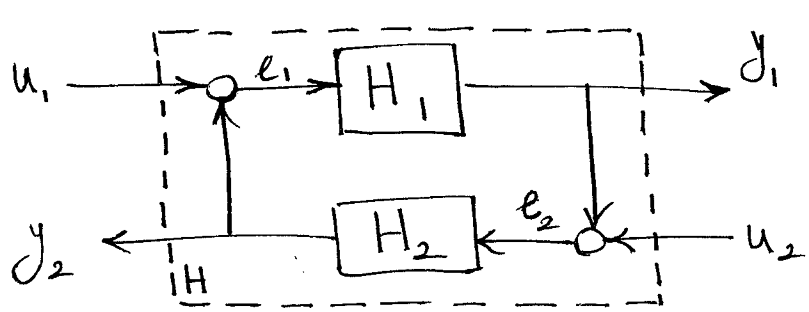
(H_∞ norm) $\longrightarrow$ peak value of Bode-mag. plot

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Then for every $\gamma > \gamma^*$, there is $P = P^T > 0$ st.

$$A^TP + PA + \frac{1}{\gamma^2} PBB^TP + CC < 0$$

* Small gain thm :



$$H : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \longrightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Suppose H_i has L_p gain $\leq \gamma_i$, $\forall i = 1, 2, \dots$

If $\boxed{\gamma_1 \gamma_2 < 1}$ then the fbk interconnection H is L_p stable.

Proof

$H_1 : \quad \|y_{1T}\|_p \leq \gamma_1 \|e_{1T}\|_p + \beta_1$

$H_2 : \quad \|y_{2T}\|_p \leq \gamma_2 \|e_{2T}\|_p + \beta_2$

$$e_1 = u_1 + y_2$$
$$e_2 = u_2 + y_1$$

This is a very general context; holds for everything

$$\begin{aligned}
\|y_{1T}\|_p &\leq \gamma_1 \|u_{1T} + y_{2T}\|_p + \beta_1 \\
&\leq \gamma_1 \|u_{1T}\|_p + \gamma_1 \|y_{2T}\|_p + \beta_1 \\
&\leq \gamma_1 \|u_{1T}\|_p + \gamma_1 \gamma_2 \|u_{2T}\|_p + \gamma_1 \gamma_2 \|y_T\|_p + \gamma_1 \beta_2 + \beta_1
\end{aligned}$$

$$\|y_{1T}\|_p \leq \frac{\gamma_1}{1-\gamma_1\gamma_2}\|u_{1T}\|_p + \frac{\gamma_1\gamma_2}{1-\gamma_1\gamma_2}\|u_{2T}\|_p + \frac{\beta_1+\gamma_1\beta_2}{1-\gamma_1\gamma_2}$$

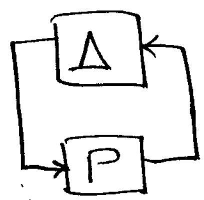
and similarly for $\|y_{2T}\|_p$

$\gamma_1\gamma_2 \neq 1 \longrightarrow$ if $\gamma_1\gamma_2 = 1$ well posedness is violated

$\gamma_1\gamma_2 > 1 \longrightarrow$ positiveness of $\mathcal{L}_p$ gains is violated

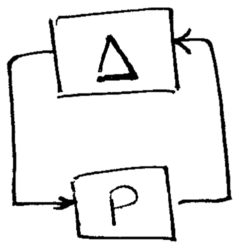
this is a sufficient condition on the stability of such interconnected systems, but it is highly conservative. Info. from phase is not taken into account!

Robust Control



Δ can be anything as long as it is norm bounded

In adaptive control the structure of uncertainty Δ is taken into account.



Δ: modeling uncertainty with L_2 gain $\leq \gamma_\Delta$

P: Plant

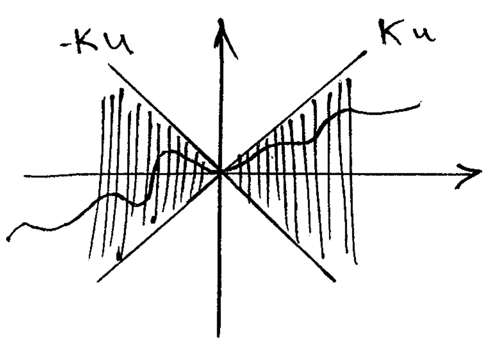
If L_2 gain $\gamma_P \leq \frac{1}{\gamma_\Delta}$

$\Rightarrow$ robust stability

What does L_p gain $\leq \gamma$ mean for a memoryless function?

$y = h(u)$

$|y| = |h(u)| \leq K \cdot |u| \quad , \quad K > 0$



bow tie!

Passivity

$$u \longrightarrow \boxed{H} \longrightarrow y$$

(same # of inputs and outputs)

$H : \mathcal{L}_{2e} \longrightarrow \mathcal{L}_{2e}$ is passive if for all $u \in L_{2e}$

and $T > 0$

$$\langle y_T, u_T \rangle_2 = \int_0^T y^T(t) u(t)\, dt \geq -\beta$$

for some $\beta > 0$.

if you don't have any initial condition $\beta = 0$.

i.e., β accounts for initial conditions and if $x(0) = 0$

$$\Rightarrow \langle y_T, u_T \rangle_2 \geq 0$$