Nonlinear Systems

Lecture 20 04/11/13

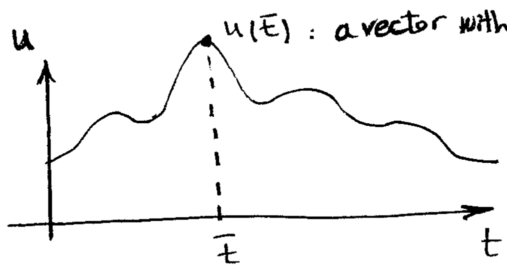
Input - output stability

$$u \longrightarrow \boxed{H} \longrightarrow y$$

u: input signal

y: output signal

H: system (mapping from inputs to output)



we can use traditional norms to get a measure of the signal at a specific time.

L_p norm for SISO systems

$$\|u\|_p^p = \int_0^\infty |u(t)|^p \, dt$$

L_p space: A signal $u \in L_p$; $p \in [1, +\infty)$ if

$$\int_0^\infty |u(t)|^p \, dt < +\infty \qquad (p \in [1, +\infty))$$

$$\sup_t |u(t)| < +\infty$$

Ex.

$$u(t) \in \mathbb{R}^m \; ; \; P = 2$$

$$\|u\|_2^2 = \int_0^\infty u^T(t)\, u(t)\, dt$$

Holder inequality:

$$\int_0^\infty |f(t)g(t)|\, dt \leqslant \|f\|_p \|g\|_q$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

special case: Cauchy Schwartz $p = q = 1$

$\|f\|_p$ signal norm

$\|f(t)\|_p$ vector norm

Minkowski:

$$\|f+g\|_p \leqslant \|f\|_p + \|g\|_p$$

Ex. $u(t) = \dfrac{1}{1+t}$; $t \geqslant 0$

$\|u\|_\infty = 1$ (achieved @ $t=0$)

Is $u \in L_1$? No! $\displaystyle\int_0^\infty \frac{dt}{1+t} = \log(t+1)\Big|_0^\infty = +\infty$

but $u \in L_2$.

Extended L_p space:

$$L_{pe} = \{ u ;\ u_T \in L_p ,\ \forall T \in [0, +\infty) \}$$

$$u_T(t) = \begin{cases} u(t) & 0 \leq t \leq T \\ 0 & \text{o.w.} \end{cases}$$

truncation operator

Ex. $\frac{1}{1+t} \in L_{1e}$ but not L_1.

Definition. An operator $H: L_{pe} \to L_{pe}$ is causal if

$$[H(u)]_T = [H(u_T)]_T \quad \forall u \in L_{pe}$$
$$\forall T \in [0, \infty)$$

Definition A causal operator $H: L_{pe} \to L_{pe}$ is L_p-stable if

$$\| y_T \|_p \leq \alpha (\| u_T \|_p) + \beta$$

class K function

constant

for all $u \in L_{pe}$ and $T \in [0, +\infty)$.

It is <u>finite gain L_p stable</u> if:

$$\|y_T\|_p \leq \underbrace{K}_{\text{constant}} \|u_T\|_p + \beta$$

The smallest possible K is called L_p gain of H.

If H is L_p stable

$$u \in L_p \Rightarrow y \in L_p$$

<u>Ex.</u>

$$\xrightarrow{u} \boxed{H(s)} \xrightarrow{y}$$

Question. When is stability in Lyapunov sense different than input-output sense.

Input-output stability $\longrightarrow$ all e-values in LHP

but there may be pole zero cancellation.

$$\left.\begin{array}{l} \text{If } Re\{\lambda_i(A)\} < 0 \\ \forall\, i = 1, \dots, n \end{array}\right\} \Rightarrow L_p \text{ stable}$$

BIBO stability

$$y(t) = \int_0^\infty h(t-\tau)\,u(\tau)\,d\tau$$

$$= \int_0^\infty h(\tau)\,u(t-\tau)\,d\tau$$

$$|y(t)| \leq \int_0^\infty |h(\tau)|\,d\tau \; \underbrace{\sup_t |u(t)|}_{L_\infty \text{ input signal}}$$

for linear systems with $h \in L_1$ for any $p \in [1, \infty]$ we have $\|y_T\|_p \leq \|h\|_1 \|u_T\|_p$

→ What is the induced L_2 gain for linear system?

H_∞ norm

this is not the tightest bound but it is when considering input-output sense ...

$\Rightarrow$ an induced L_p gain $\leq \|h\|_1$

Tight bound for $p = \infty$

(Note: an induced L_2 norm is H_∞ norm : $\sup_\omega |H(j\omega)|$; SISO

$\sup_u \sigma_{max}(H(j\omega))$; MIMO

L_p - stability of state-space models :

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \qquad \circledast$$

Lyapunov-like conditions for L_p stability :

(Khalil ; Thm 5.1)

If the following two assumptions hold, then the system $\circledast$ is finite gain, L_p stable for any $p \in [1, \infty)$

A1) There is Lyapunov function, $V(x)$ st.

$$\text{stability} \left\{ \begin{aligned} & c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \\ & \frac{\partial V}{\partial x} f(x, \underset{\substack{\uparrow \\ u}}{0}) \leq -c_3 \|x\|^2 \end{aligned} \right.$$

$$\| \underbrace{\frac{\partial V}{\partial x}}_{\nabla V} \| \leq C_4 \|x\|$$

A2) $\|f(x,u) - f(x,0)\| \leq C_5 \|u\|$

$\|h(x,u)\| \leq C_6 \|x\| + C_7 \|u\|$

for some constants $C_i > 0 \; ; \; i = 1, \dots, 7$

Ex.

$\dot{x} = -x - x^3 + u$

$y = \tanh(x) + u$

if u is zero $\longrightarrow$ system is stable

A1) holds with $V(x) = \frac{1}{2}x^2$ as a Lyapunov function

A2) $\|f(x,u) - f(x,0)\| = \|u\|$ holds with $C_5 = 1$

$$|\tanh(x) + u| \leq |\tanh(x)| + |u| \leq |x| + |u|$$

$C_6 = 1 \quad C_7 = 1$

Ex. $\dot{x} = -x + x^2 u$

$y = x$

A1) $u = 0 \longrightarrow$ stable ✓ $\quad \begin{array}{l}\dot{x} = -x \\ y = x\end{array}$

but what about input output case?

Choose $u = \text{const.} \Rightarrow$ finite escape time

$\Rightarrow$ not L_p stable!

A2) $|-x + x^2 u - (-x) - x^2 \cdot 0| = |x^2 u| = \underbrace{|x^2|}|u|$

$C_5 = C_5(x)$

no uniform const. can be found!

★ A state-space characterization of L_2 stability

$$\left.\begin{array}{l}\dot{x} = f(x) + g(x)\,u \\ y = h(x)\end{array}\right\}$$ For input affine systems can be more specific

If there is a cts differentiable function $V(x)$ st.

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} \frac{\partial V}{\partial x} g(x) g^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

then the system has an L_2 gain of γ.

no Lyapunov functions here $\longrightarrow$ we will call V functions that hold in such conditions storage functions.