Nonlinear Systems

Lecture 18

04/02/13

Last time

Ex 1 $y = \psi^T \theta$

ψ: regressor; θ: unknown constant parameters

Convergence of gradient-based parameter estimator

Ex 2

$$\dot{x}_1 = -a x_1 + W^T(t) x_2$$

$$\dot{x}_2 = -W(t)\, x_1$$

$W(t)$: persistently exciting $\oplus$ $a > 0$ $\Rightarrow$ uniform asymptotic stability of the origin

Example Model Reference Adaptive Control (MRAC)

$$\dot{y} = a y + u \quad \cdots (1)$$

$u(\cdot)$	input signal	scalars	$u(t) \in \mathbb{R}$
$y(\cdot)$	output signal		$y(t) \in \mathbb{R}$

a: constant but unknown parameter

Reference Model :

$$\dot{y}_m(t) = -a_m y_m(t) + r(t) \quad \cdots \quad (2)$$

$a_m > 0$ (fixed)

$r(\cdot)$: reference signal

If a was known, $u(t) = -a y(t) - a_m y(t) + r(t)$

$$= -(a + a_m)\, y(t) + r(t)$$

$$= -k\, y(t) + r(t) \qquad (3)$$

would give us (3) $\rightarrow$ (1)

$$\dot{y}(t) = -a_m y(t) + r(t) \qquad (4)$$

Error variable : $e(t) = y(t) - y_m(t)$

(4) $-$ (2) $\Rightarrow$

$$\dot{e}(t) = -a_m e(t) \Rightarrow \boxed{e(t) = \exp(-a_m t)\, e(0)}$$

Summary: Works well when a is known

Challenge: What to do when a is unknow

Idea: use,

$$\boxed{u(t) = -\hat{K}(t)\, y(t) + r(t)} \qquad (5)$$

$\dot{\hat{K}}(t)$ to be determined

Write:

$$K = \hat{K}(t) + \underbrace{\tilde{K}(t)}_{\text{estimation error}}$$

$$\hat{K}(t) = K - \tilde{K}(t) \qquad (6)$$

(6) $\rightarrow$ (5) : objective: find $\dot{\hat{K}} = -\dot{\tilde{K}}$?

$$u(t) = -K y(t) + \tilde{K}(t) y(t) + r(t) \qquad (7)$$

(7) $\longrightarrow$ (1)

$$\dot{y}(t) = -a_m y(t) + \tilde{K}(t) y(t) + r(t) \qquad (8)$$

(8)$-$(2) $\Rightarrow$ $\dot{e}(t) = -a_m e(t) + y(t)\tilde{K}(t)$

$\dot{\tilde{K}}(t) = ?$

We need to find $\dot{\tilde{K}}(t)$ s.t. $\begin{bmatrix} e \\ \tilde{K} \end{bmatrix}$ system is UAS!

Uniformly Asymptotically Stable

Propose a lyapunov function:

$$V(e,\tilde{K}) = \frac{1}{2}e^2(t) + \frac{1}{2}\tilde{K}^2(t)$$

$$\dot{V} = e \cdot \dot{e} + \tilde{K} \cdot \dot{\tilde{K}} = e\left[-a_m e + y \cdot \tilde{K}\right] + \tilde{K}\dot{\tilde{K}}$$

$$= -a_m e^2 + \tilde{K}\left[\dot{\tilde{K}} + e \cdot y\right]$$

choose $\dot{\tilde{K}}(t) = -e(t)y(t)$

$$\Rightarrow \boxed{\dot{V}(e,\tilde{K}) = -a_m e^2}$$

1)* negative semi-definite

$\left(\dot{V} = -a_m e^2 - 0\tilde{K}^2 \leq 0\right)$

$\Rightarrow$ US

$$\dot{e} = -a_m e + y(t)\tilde{K}$$

$$\dot{\tilde{K}} = -y(t)e + 0 \cdot \tilde{K}$$

2)* If $r(t)$ is bounded $\Rightarrow e(t) \xrightarrow{t \to 0} 0$ Why?

↳ Barbolat's lemma allows you to conclude this.

$$\begin{cases} \dot{e} = -a_m e + y(t)\tilde{K} \\ \dot{\tilde{K}} = -y(t)e + 0.\tilde{K} \end{cases}$$

$$\dot{x}_1 = -a_m x_1 - W(t)x_2$$

$$\dot{x}_2 = W(t)x_1$$

3)* $y(t)$: persistently exciting $\Rightarrow$ UAS

Output of the system depends on $r(t)$

$$y(t) = e(t) + y_m(t)$$

$$\dot{y}_m(t) = -a_m y(t) + \underline{\underline{r(t)}}$$

So based on this even though $r(t)$ was just considered to be bounded (no need to be rich enough) at the end $\tilde{K} \xrightarrow{t\to\infty} K$

Summary:

$$u(t) = -\hat{K}(t)\underbrace{y(t)}_{\text{measured}} + r(t) \rightarrow \text{given reference}$$

and ...

$$\dot{\hat{K}}(t) = y(t)\left(y(t) - y_m(t)\right)$$

Aside

$$\tilde{K}(t) = \underbrace{K}_{\text{const. unknown}} - \hat{K}(t) \Rightarrow \dot{\tilde{K}}(t) = -\dot{\hat{K}}(t)$$

therefore the control will be: (Implementation)

$$\begin{cases} u(t) = -\hat{K}(t)\, y(t) + r(t) \\ \dot{\hat{K}}(t) = y(t)\left(y(t) - y_m(t)\right) \\ \dot{y}_m(t) = -a_m y_m(t) + r(t) \end{cases}$$

Dynamical controller

Backstepping

Consider

$$\dot{x} = \underbrace{f(x)}_{\text{vector field}} + \underbrace{g(x)}_{\text{vector field}} \cdot u$$

u → scalar control input

$x(t) \in \mathbb{R}^n$

Assume that there is

$$u = \alpha(x)$$

st. with $V(x)$: pos. definite radially unbounded

$$\frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x)\right] \leq -W(x) < 0 \Rightarrow \text{GAS}$$

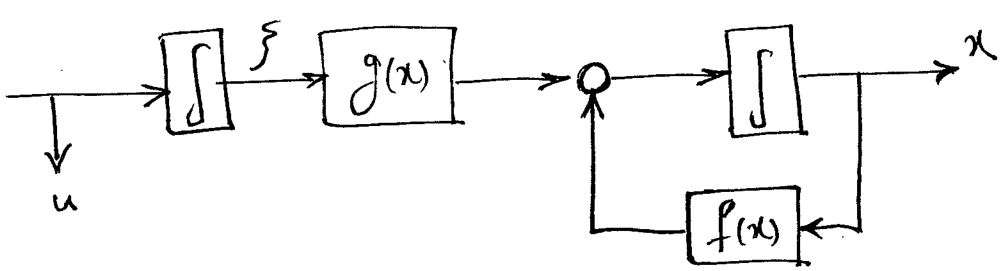
Now consider

$$\dot{x} = f(x) + g(x)\xi \qquad (\mathrm{I})$$

$$\dot{\xi} = u \qquad (\mathrm{II})$$

with $\xi = \alpha(x)$ satisfying the above conditions.

Q. Can we design $\xi = \beta(x, \xi)$ st. the origin of (I), (II) is GAS?



ξ : not control

Think of it as a virtual control, penelize the difference between ξ, and $\alpha(x)$ and figure out u so that we guarantee GAS of (I), (II)

Augment $V(x)$

$$\frac{\partial V}{\partial x}[f(x)+g(x)\alpha(x)] = -W(x) < 0$$

with $(\xi-\alpha(x))^2$

$$V_a(x,\xi) = V(x) + \frac{1}{2}(\xi-\alpha(x))^2$$

$$Z = \xi - \alpha(x)$$

$$\dot{z} = \dot{\xi} - \frac{\partial\alpha}{\partial x}\dot{x} = u - \frac{\partial\alpha}{\partial x}(f(x)+g(x)\xi)$$

$$V_a(x,z) = V(x) + \frac{1}{2}z^2$$

$$\dot{V}_a = \dot{V} + z\dot{z} = \frac{\partial V}{\partial x}[f(x)+g(x)\underbrace{(\alpha(x)+Z)}_{\xi}] +$$

$$+ Z\left[u - \frac{\partial\alpha}{\partial x}(f(x)+g(x)\xi)\right]$$

$$\dot{V}_a = \underbrace{\frac{\partial V}{\partial x}\left[f(x) + g(x)\alpha(x)\right]}_{-W(x)} + z\left[u - \frac{\partial \alpha}{\partial x}\left[f(x) + g(x)\xi\right] + \frac{\partial V}{\partial x} g(x)\right]$$

$$-W(x) \quad + $$

Since we can measure z there is no need to set this to zero!

$$-Kz^2$$

Made possible by:

$$u = \frac{\partial \alpha}{\partial x}\left[f(x) + g(x)\xi\right] - \frac{\partial V}{\partial x} g(x) - Kz$$

$$K > 0$$