Nonlinear Systems

Lecture 13 03/07/13

Today :

- Comparison Functions
  (preparation for time-varying systems)
- Stability of time-varying systems
- Lapunov based characteristics

* Comparison functions

3 types : $K$, $K_\infty$, $KL$

$K$ : A cts. function $\alpha : [0,\infty) \to [0,\infty)$
is a class-$K$ function if : $\alpha(0) = 0$
it is strictly increasing

If in addition to these two properties :

$$\alpha(r) \xrightarrow{r \to +\infty} +\infty$$

we'll say that $\alpha$ is a $K_\infty$ function.

Ex1

$\alpha(r) : r^c ; \quad c>0$

$\alpha(0) = 0$

$\frac{d\alpha}{dr} = c r^{c-1} > 0$

$\Rightarrow$ class K

$\Rightarrow \alpha$ is increasing

Also, in addition to being of class K since

$r^c \xrightarrow{r\to\infty} \infty \quad \Rightarrow \alpha(r)$ is class $K_\infty$.

Ex 2

$\alpha(r) = \tan^{-1}(r)$

$\alpha(0) = 0$

$\frac{d\alpha}{dr} = \frac{1}{1+r^2} > 0$

class-K

but not $K_\infty$. (defined only between $[0, \pi/2)$)

- A cts function $\beta : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ is class KL if:

a) $\beta(\cdot, s)$ : is a class-K (for every fixed s)

b) $\beta(r, \cdot)$ : is decreasing and $\beta(r,s) \xrightarrow{s\to\infty} 0$ (for any fixed r)

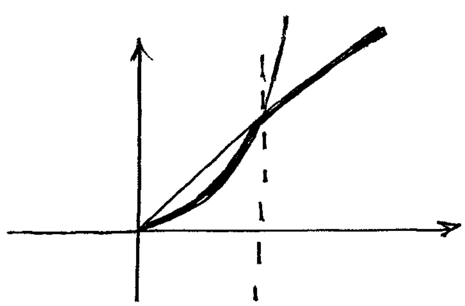
**Ex3** $\beta(r,s) = K r^{c} e^{-as}$ $\quad K, a > 0$, $\quad c > 0$

**Ex4** $\beta(r,s) = \dfrac{r}{Krs+1}$ $\quad K > 0$

for fixed $s$:

$$\left.\begin{aligned} &\beta(0,s) = 0 \\ &\frac{d\beta}{dr} = \frac{\cancel{Krs}+1-\cancel{rKs}}{(Krs+1)^2} > 0 \end{aligned}\right\} \begin{array}{l}\beta(\cdot,s) \\ \text{class } K.\end{array}$$

**Ex5** $\alpha(r) = \min(r, r^2)$



class $K$-$\infty$ function

Moral: We don't need function to be $C^1$ in order to be $K$ or $K_\infty$.

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Want to study stability of $\quad \dot{x} = f(x,t)$, $\quad f(0,t) = 0 \qquad (*)$

Time invariant system

$$\dot{z} = g(z) \qquad (1)$$

Consider a trajectory

$$\dot{\bar{Z}}(t) = g(\bar{Z}(t)) \qquad (2)$$

Change of coordinates :

$$x(t) = z(t) - \bar{Z}(t) \qquad (3)$$

$$\frac{dx(t)}{dt} = \dot{z}(t) - \dot{\bar{Z}}(t) = g(z(t)) - g(\bar{z}(t))$$
$$= g(x + \bar{Z}(t)) - g(\bar{z}(t))$$
$$=: f(t, x)$$

$$\underbrace{z(t)}_{\text{original state}} = \underbrace{\bar{Z}(t)}_{\text{time varying trajectory}} + \underbrace{x(t)}_{\text{fluctuations}}$$

Choice of initial time matters $(t_0)$

The origin of (*) is stable if for every $\varepsilon > 0$, there is $\delta(\varepsilon, t_0) > 0$ st. $\|x(t_0)\| < \delta(\varepsilon, t_0) \Rightarrow \|x(t)\| < \varepsilon$ for all $t \geq t_0$

trajectory that starts @ $x(t_0)$ : $\phi(t, x(t_0))$

If in the above definition $\delta = \delta(\epsilon)$ [i.e., independent of $t_0$]

then the origin is <u>uniformly</u> stable.

Ex

$$\dot{x} = \frac{-x}{1+t} = a(t)\cdot x$$

$$\frac{dx}{dt} = -a(t)\,dt$$

$$x(t) = x(t_0)\,e^{-\int_{t_0}^{t}\frac{1}{1+\tau}\,d\tau} = x(t_0)\,e^{\log\frac{1+t_0}{1+t}}$$

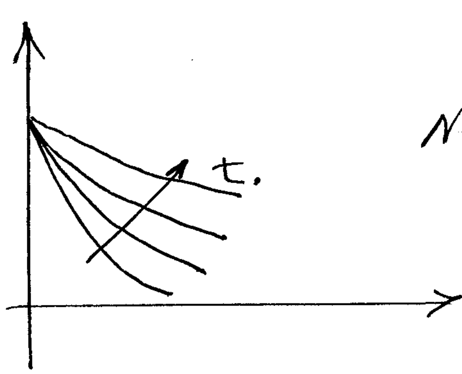
$$\Rightarrow \quad x(t) = x(t_0)\,\frac{1+t_0}{1+t}$$

Since $t \geq t_0 \Rightarrow \dfrac{1+t_0}{1+t}$ decreasing

$\Rightarrow$ uniformly stable.

$$x(t) = x(t_0)\,\frac{1}{\frac{1}{1+t_0} + \frac{t}{1+t_0}} \longrightarrow x(t) = x(t_0)\,\frac{1}{\frac{1+t_0-t_0+t}{t_0+1}}$$

$$= x(t_0)\,\frac{1}{1+\frac{t-t_0}{1+t_0}}$$




Note! absence of a uniform rate of convergence.

It is more convenient to define stability of time varying systems using comparison functions.

The origin of ⊛ is (a) uniformly stable if there is a class K function $\alpha(\cdot)$ and a constant $c>0$ st.

$$\|x(t)\| \leq \alpha(\|x_0\|) \text{ for all } t \geq t_0 \text{ and all } x(t_0)$$

st. $\|x(t_0)\| < c$.

(b) uniformly asymptotically stable if

there is a class KL function $\beta(\cdot,\cdot)$ and a constant $c>0$

st. $\|x(t)\| \leq \beta(\|x(t_0)\|, t-t_0)$ for all $t \geq t_0$

and for all $x(t_0)$ st. $\|x(t_0)\| \leq c$.

If in addition: $\beta(r,s) = Kre^{-as}$; $K>0, a>0$ then the origin is uniformly exponentially stable

* Lyapunov functions for time-varying systems $\dot{x} = f(x,t)$

$V(x,t)$.

$$\dot{V}(x,t) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(x,t)$$

**Fact:** If $V(x)$ is positive definite, then we can find class $K$ functions $\alpha_1$ and $\alpha_2$ s.t.

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

in addition if $V(x)$ is radially unbounded then we can choose $\alpha_1(\cdot)$ to be $K_\infty$.

**Ex** $\quad V(x) = x^T P x$

if $P = P^T > 0$

$$\lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2$$

$$\alpha_1(r) = \lambda_{min}(P) r^2$$

$$\alpha_2(r) = \lambda_{max}(P) r^2$$

**Note!** In linear case:

$$\dot{x}(t) = A(t)x(t)$$

$$x(t) = \Phi(t,t_0)x(t_0)$$

$\Phi(t,t_0)$: state transition matrix

$$\|x(t)\| \leq \|\Phi(t,t_0)\| \|x(t_0)\| \leq K\|x_0\| e^{-a(t-t_0)}$$

Exponential stability of time varying linear system $\Leftrightarrow$ $\|\Phi(t,t_0)\| \leq K e^{-a(t-t_0)}$