Nonlinear Systems

Lecture 09 02/19/13

Last time:

Existence & uniqueness of solutions to $\dot{x} = f(t,x)$; $x(t_0) = x_0$

If f is piecewise cts in t, and

- cts in x $\Rightarrow$ existence for $t \in [t_0, t_f)$
- locally Lipschitz in x $\Rightarrow$ existence and uniqueness for $t \in [t_0, t_f)$
- globally Lipschitz in x $\Rightarrow$ " for $t \in [t_0, \infty)$

cts dependence on IC's

$x_1(t), x_2(t)$: 2 sol'ns of $\dot{x} = f(t,x)$
starting from x_{10} & x_{20}, and staying in a set with Lipschitz constant L for $t \in [0,T)$

$\forall \varepsilon > 0 \ \exists \ \delta(\varepsilon, T) > 0$ st.

$$\| x_{10} - x_{20} \| < \delta \Rightarrow \| x_1(t) - x_2(t) \| \leq \varepsilon \quad \text{for all } t \in [0,T)$$

Today:

Sensitivity w.r.t. parameters

Lyapunov stability

How about cts dependence w.r.t. parameters?

$$\dot{x} = f(t, x, \mu) \qquad (1)$$

μ: constant parameter

Augment differential eq'n (1) with $\dot{\mu} = 0 \qquad (2)$

and study system (1)(2) with $z = \begin{bmatrix} x \\ \mu \end{bmatrix}$

$$\dot{z} = g(t, z)$$

$$g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} f(t, x, \mu) \\ 0 \end{bmatrix}$$

Using cts dependence on IC's $\Rightarrow$ cts dependence on parameters

* Sensitivity of solutions w.r.t parameters:

Given $\dot{x} = f(t, x, \mu)$

f is cts in all parameters and also cts diff.ble in the vicinity of $\bar{\mu}$.

For given $\bar{\mu}$ $\Rightarrow$ existence and uniqueness on $[t_0, t_f)$

let the corresponding trajectory be $x(t, \bar{\mu})$

Based on cts dependence on ICs and parameters

$\Rightarrow$ there is a sol'n around $\bar{\mu}$. We want to study how sol'n would change with changes in μ.

Write sol'n to,

$$\dot{x} = f(t, x, \mu)$$

$$x(t,\mu) = x_0 + \int_{t_0}^{t} f(t, x(\tau,\mu), \mu)\, d\tau \qquad (*)$$

Differentiate $(*)$ w.r.t. μ

$$\frac{\partial x(t,\mu)}{\partial \mu} = \underbrace{\frac{\partial x_0}{\partial \mu}}_{0} + \int_{t_0}^{t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial \mu} + \frac{\partial f}{\partial \mu} \right) d\tau$$

Introduce notation:

$$x_\mu(t,\mu) = \frac{\partial x(t,\mu)}{\partial \mu}$$

$$x_\mu(t,\mu) = \int_{t_0}^{t} \left(\frac{\partial f(\tau, x, \mu)}{\partial x} x_\mu(\tau, \mu) + \frac{\partial f}{\partial \mu}(\tau, x, \mu) \right) d\tau$$

Differentiate w.r.t. time :

$$\dot{x}_\mu(t,\mu) = \frac{\partial f(t, x(t,\mu),\mu)}{\partial x} \cdot x_\mu(t,\mu) + \frac{\partial f}{\partial \mu}(t, x(t,\mu),\mu)$$

Let $S(t) := \left.\dfrac{\partial x(t,\mu)}{\partial \mu}\right|_{\bar{\mu}}$ similarly : $A(t) = \left.\dfrac{\partial f(t,x(t,\mu),\mu)}{\partial x}\right|_{\substack{x = x(t,\bar{\mu}) \\ \mu = \bar{\mu}}}$

$$B(t) := \left.\frac{\partial f(t,x(t,\mu),\mu)}{\partial \mu}\right|_{x = x(t,\bar{\mu})\,;\,\mu = \bar{\mu}}$$

$$\Downarrow$$

$$\dot{S}(t) = A(t)\cdot S(t) + B(t)$$

Linear differential equations in the elements of sensitivity matrix $S(t)$

Ex. $x(t) \in \mathbb{R}^2$

$\mu \in \mathbb{R}^3$

$$\begin{bmatrix} x_{1\mu_1} & x_{1\mu_2} & x_{1\mu_3} \\ x_{2\mu_1} & x_{2\mu_2} & x_{2\mu_3} \end{bmatrix}$$

Given $x(t,\bar{\mu})$

$$x(t,\mu) = x(t,\bar{\mu}) + \left.\frac{\partial x(t,\mu)}{\partial \mu}\right|_{\bar{\mu}} (\mu-\bar{\mu}) + \text{HOT}$$

$$\Downarrow$$

$$x(t,\mu) \approx x(t,\bar{\mu}) + S(t)\,(\mu-\bar{\mu})$$

$\hookrightarrow + O(\|\mu\|^2)$

$$\dot{x} = f(t,x,\bar{\mu}) \quad ; \quad x(t_0) = x_0$$

$$\dot{S}(t) = A(t)\,S(t) + B(t)$$

$A(t)$, $B(t)$: functions of $x(t,\bar{\mu})$

Ex1

Fold bifurcation $\qquad \dot{x} = x^2+\mu$

$$f(x,\mu) = x^2+\mu$$

$$\partial f/\partial x = 2x \quad ; \quad \partial f/\partial \mu = 1$$

$$\begin{cases} \dot{x} = x^2+\bar{\mu} & x(0)=x_0 \\ \dot{S} = 2xS+1 & S(0)=0 \end{cases}$$

one way coupling

$$\dot{S}(t) = 2\bar{x}(t)\,.\,S(t) + 1 \quad ; \quad S(0) = 0$$

$\bar{x}$, fixed trajectory of the original system $\quad \dot{x} = x^2 + \bar{\mu}$

the sol'n can be explored by integrating rhs. given $\bar{x}$ being a state transition matrix.

Ex2

(Ex 3.7 Khalil)

$$\dot{x}_1 = x_2 \qquad\qquad\qquad\qquad\qquad = f_1(x_1, x_2, \mu)$$

$$\dot{x}_2 = -c\underset{\sin(x_1)}{\cancel{}} - (a + b\cos x_1)x_2 = f_2(x_1, x_2, \mu)$$

$$\mu = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \bar{\mu} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \underset{-c\cos x_1}{\cancel{}} + \underset{0}{\cancel{b}}(\sin x_1)x_2 & -(a + \underset{0}{\cancel{b}}\cos x_1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -\cos\bar{x}_1(t) & -1 \end{bmatrix} = A(t)$$

$$B(t) = \frac{\partial f}{\partial \mu} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial f_2}{\partial a} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial c} \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} 0 & 0 & 0 \\ -\bar{x}_2(t) & -\bar{x}_2(t)\cos(\bar{x}_1(t)) & -\sin(\bar{x}_1(t)) \end{array}\right]$$

$$\dot{S}(t) = A(t)\,S(t) + B(t) \quad ; \quad S(0) = 0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$S(t) = \begin{bmatrix} x_{1a} & x_{1b} & x_{1c} \\ x_{2a} & x_{2b} & x_{2c} \end{bmatrix}$$

$$x_{i\alpha} = \frac{\partial x_i}{\partial \alpha} \qquad \begin{array}{l} i = 1, 2 \\ \alpha = a, b, c \end{array}$$

see plots in page 102.

x_{1c} and x_{2c} show the most changes and amplitudes sensitivity on parameter c is the most.

Lyapunov Stability:

for now consider time invariant systems:

$$\dot{x} = f(x)$$

W.L.O.G. assume e.p. @ the origin

$$\bar{x} = 0 \Rightarrow f(0) = 0$$

if $\bar{x} \neq 0$ with $f(\bar{x}) = 0$ do a change of coordinates

$$Z(t) = x(t) - \bar{x} \Rightarrow x = z + \bar{x}$$

$$\downarrow$$

$$x(t) = Z(t) + \bar{x} \Rightarrow \dot{x}(t) = \dot{z}(t)$$

$$f(z+\bar{x}) = 0 \Rightarrow z = 0$$

$$\boxed{\dot{z} = f(z+\bar{x})}$$

e.p. $z = 0$