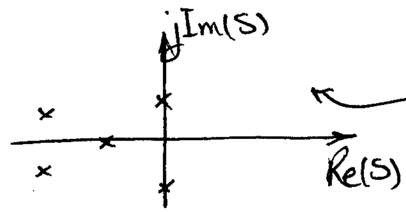
Nonlinear Systems

Lecture 08 02/14/13

Last time:

Center manifold theory, $\dot{x} = f(x) \leftarrow$ original system



$\leftarrow$ Spectrum of linearization around e.p.

- K e-values on $j\omega$ axis
- $(n-k)$ e-values in the LHS

Then,

. Stability properties characterized by stability properties of the reduced order system.

$$\dot{y} = A_1 y + g_1(y, \underbrace{h(y)}_{\text{center manifold}})$$

$$z = h(y)$$

$$\begin{cases} \dot{y} = A_1 y + g_1(y,z) \\ \dot{z} = A_2 z + g_2(y,z) \end{cases}$$

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \quad \begin{bmatrix} y \\ z \end{bmatrix} = Tx$$

Today:
- existence & uniqueness of sol'ns
- continuous dependence on IC / parameters
- Sensitivity of sol'ns wrt. changes in parameters.

Existence and uniqueness

$$\dot{x} = f(t, x) \quad ; \quad x(t_0) = x_0$$

in time varying case

In 5231 we restricted $f(t,x)$ to $f(t,x) = A(t)x$
or $f(x) = Ax$

in time invariant case

piecewise continuity was all we needed for existence & uniqueness in the Linear case.
But what can be done for the nonlinear case?

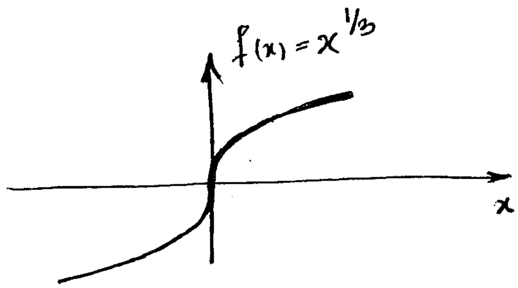
We'll assume that $f(t,x)$ is piece-wise cts function of time and discuss conditions on the dependence on x that guarantee existence and uniqueness.

Ex. $\dot{x} = x^{1/3}$ $\quad x(0) = x_0$

for $x_0 = 0$ we have 2 solutions : $\boxed{x(t) \equiv 0}$ one sol'n

another sol'n: $x^{-1/3} dx = dt$

$$\tfrac{3}{2} x^{2/3} \Big|_0^{x(t)} = t \Big|_0^t \Rightarrow \boxed{x(t) = \left(\frac{2t}{3}\right)^{3/2}}$$



$$\frac{\partial f}{\partial x}\Big|_{x=0} = x^{-2/3}\Big|_{x=0} = \infty$$

Therefore, if f is a cts function of x

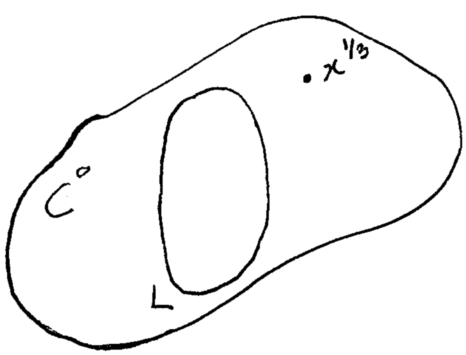
$$\not\Downarrow$$

existence & uniqueness

(problem: infinite slop of f at the origin)

Fact: If $f(t,x)$ is cts in $x \Rightarrow$ there is a solution on $[0, t_f)$ (but it may not be unique, see Ex. 1)

$\therefore$ We need to further restrict classes of admissable f's



Lipschitz continuity

$$\|f(t,x) - f(t,y)\| \leq L\,\|x-y\|$$

If this holds for all t ~~and for all t~~ and for all points in a certain nhd of an arbitrary point $\bar{x}$ in $\mathbb{R}^n$, for some L [Lipschitz constant] then f is locally lipschitz [in x]

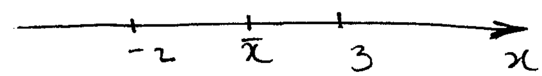
If this holds for any $x, y \in \mathbb{R}^n$ then f is globally Lipschitz.

Ex 2

$$f(x) = x^2$$

$$f(x) - f(y) = x^2 - y^2 = (x+y)(x-y)$$

$$|f(x) - f(y)| \leq |x+y||x-y|$$



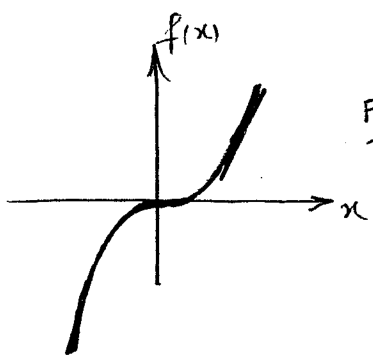
$\therefore$ $f(x) = x^2$ is locally Lipschitz
but not globally Lipschitz

Ex 3

$$f(x) = x^3$$

$$x^3 - y^3 = \underbrace{(x^2 + xy + y^2)}(x-y)$$

Same conclusion as in (Ex. 2) (local but not global)



Fact: Any function that is continuously differentiable (differentiable & has its 1st derivative) is locally Lipschitz.

<u>Ex1</u>

$$f(x) = x^{1/3}$$

$$\partial f/\partial x = \frac{1}{x^{2/3}} \Rightarrow \text{not cts diff.}$$

<u>Ex2</u>

$$f(x) = x^2 \Rightarrow \partial f/\partial x = 2x$$

<u>Ex3</u>

$$f(x) = x^3 \Rightarrow \partial f/\partial x = 3x^2$$

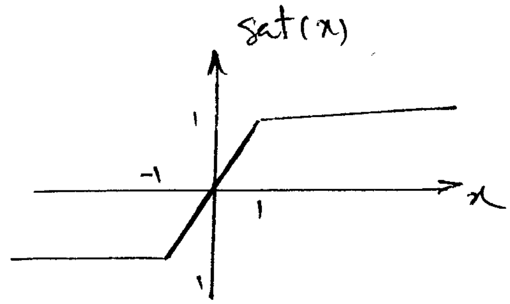
the 2 last examples are cts diff'ble $\Rightarrow$ locally Lipschitz
but $\partial f/\partial x$ is not uniformly bounded in x $\Rightarrow$ not globally Lipschitz

Summary: <u>Sufficient</u> conditions for being Lipschitz is C^1.

$$f \in C^1 \Rightarrow \text{loc. Lipschitz}$$

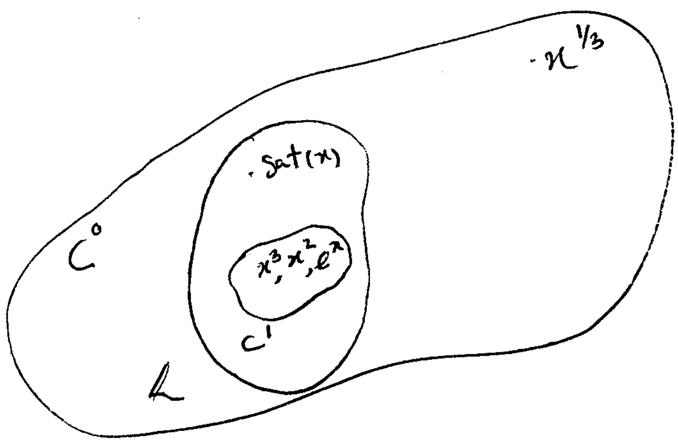
$$\left\| \partial f/\partial x \right\| \leq L \text{ forall } x \in \mathbb{R}^n \Rightarrow \text{globally Lipschitz}$$

Ex 4:



$f(x) = sat(x)$ is not differentiable (@ $x = \pm 1$ problamatic)

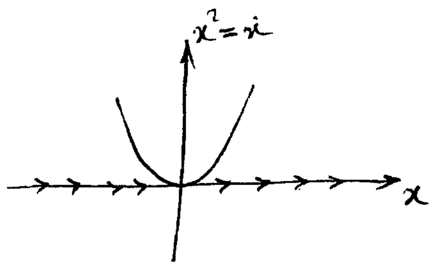
yet globally Lipschitz (with $L = 1$)

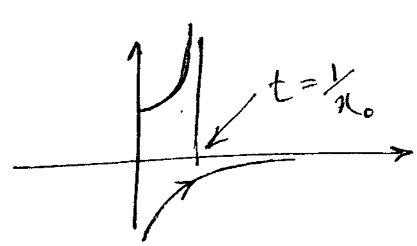


Back to Ex2: $\dot{x} = x^2 \longrightarrow x^{-2}dx = dt \rightarrow -x^{-1}\Big|_{x_0}^{x(t)} = t - 0$

$\ldots \Rightarrow \quad x(t) = \dfrac{x_0}{1 - x_0 t}$

so $x(t) = \dfrac{1}{1-t}$ when $x_0 = 1$ *finite escape time!*





Here we have existence and uniqueness on $[0, t_f)$

finite time interval

Note! solutions blow up in finite time (finite escape time)

Summary! f is piecewise cts w.r.t. time

of sufficient conditions

1) f cts w.r.t. $x \Rightarrow$ existence on $[0, t_f)$

2) f locally Lipschitz w.r.t $x \Rightarrow$ existence & uniqueness on $[0, t_f)$

3) f globally Lipschitz w.r.t. $x \Rightarrow$ existence & uniqueness on $[0, \infty)$

proof $\rightarrow$ Khalil.

Ex 5 $\dot{x} = -x^3$

cts differentiable : $\frac{\partial f}{\partial x} = -3x^2$

but not globally Lipschitz. yet, existence and uniqueness on $[0, +\infty)$

proof solve differential eq'n

Back to Linear systems :

Ex 6

$$\dot{x} = A(t)\,x$$

$$\|f(t,x) - f(t,y)\| = \|A(t)x - A(t)y\| = \|A(t)(x-y)\| \leq \|A(t)\|\,\|x-y\|$$

induced norm

A piecewise cts $\rightarrow$ globally Lipschitz

Fact

If existence and uniqueness $\Rightarrow$ cts dependence on initial conditions on finite time intervals.

Given f : locally Lipschitz

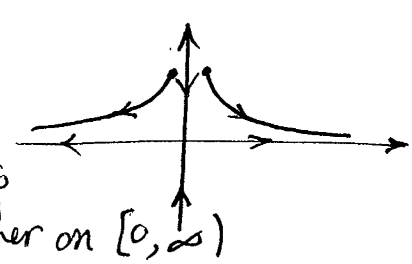
$\forall \varepsilon > 0$ and $T \in [0, t_f)$ there is $\delta = \delta(\varepsilon, T)$ st.

$$\|x_0 - y_0\| \leq \delta \Rightarrow \|\phi(t, x_0) - \phi(t, y_0)\| < \varepsilon$$

for $t \in [0, T)$

Moral : Even if f is globally Lipschitz this cannot be extended on $[0, +\infty)$

Ex Saddle points



even two initial pts that are close to each other get infinitly far from each other on $[0, \infty)$