Nonlinear Systems

Lecture 05 02/05/13

last time:

Phase portraits of 2nd order systems
Hartman-Grobman Thm
Bendixon Thm (absence of periodic orbits in 2nd order systems)

Today:

Application of Bendixon Thm
Poincare-Bendixon Thm (existence of periodic orbits in 2nd order systems)

Hopf Bifurcation ($\begin{smallmatrix} \text{super} \\ \text{sub} \end{smallmatrix} >$ critical)

Bendixon: D simply connected domain (region w/o holes)

dynamics:
$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}\right\} \quad x_i(t) \in \mathbb{R}$$

· $\operatorname{div} f$: not identically zero AND doesn't change sign in D

$$\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

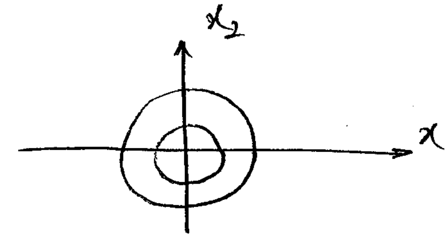
$\Rightarrow$ No periodic orbits in D

Ex 1 Given $A \in \mathbb{R}^{2\times 2}$ unless $trace(A)=0 \Rightarrow$ no periodic orbits

$$\dot{x}_1 = a x_1 + b x_2 \qquad \Longleftrightarrow \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \; ; \quad \left.\begin{array}{l} \dfrac{\partial f_1}{\partial x_1} = a \\ \dfrac{\partial f_2}{\partial x_2} = d \end{array}\right\} \; \operatorname{div} f = a + d = trace(A)$$
$$\dot{x}_2 = c x_1 + b x_2$$

Note! In general GAS of a unique e.p. outrules the presence of any periodic orbit.

$$A = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix} \qquad \lambda_{1,2}(A) = \pm j\beta$$

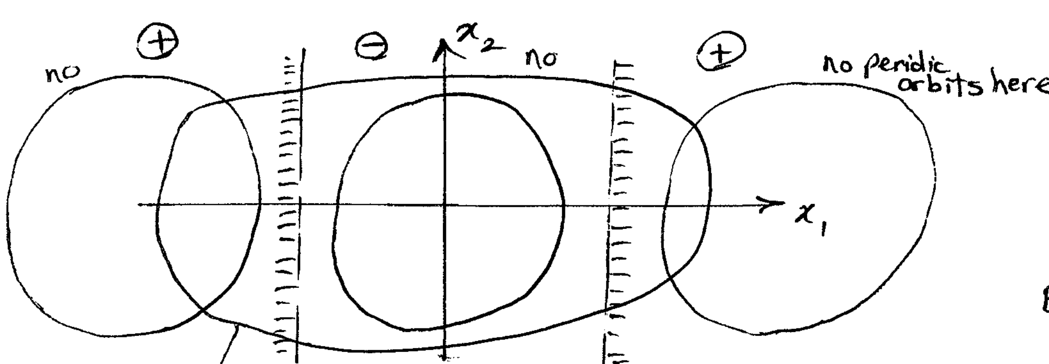


Ex 2

$$x_1 = x_2 = f_1$$
$$\dot{x}_2 = -\alpha x_2 + x_1 - x_1^3 + x_1^2 x_2 = f_2$$

$\alpha > 0$

$$\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0 - \alpha + x_1^2 \Rightarrow \operatorname{div} f = x_1^2 - \alpha$$



Bendixon inconclusive

can't use Thm in this region (we don't know about periodic orbits)

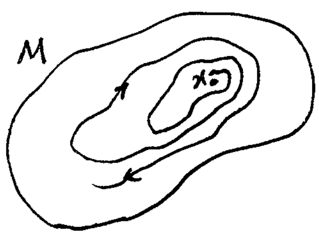
Bendixon can only tell you about the absence of periodic orbits. It cannot tell you if you have them, it can only rule their presence.

Aside

Invariant Sets

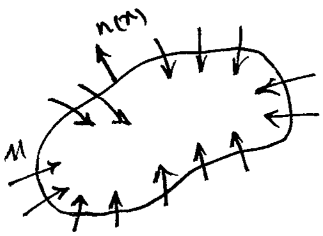
$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

Trajectory starting at x_0 will be denoted by $\phi(t, x_0)$



A set M is positively (negatively) invariant if $x_0 \in M \Rightarrow \phi(t, x_0) \in M$, $\forall t \geq 0$ $(\forall t \leq 0)$

So what condition should be satisfied for this to happen?



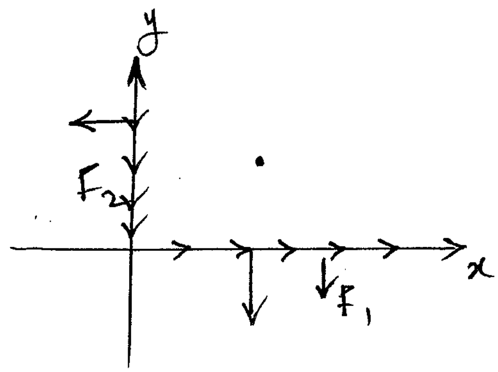
$f(x)$ should always pointing into the set.

Ex 1 Predator–Prey model

$$\dot{x} = (a - by)x \qquad \text{prey}$$

$$\dot{y} = (cx - d)y \qquad \text{predator}$$

a, b, c, d positive constants , xy: chance of encounter



$f(x)\cdot n(x) = 0$

$\Rightarrow$ first quadrant is positively invariant

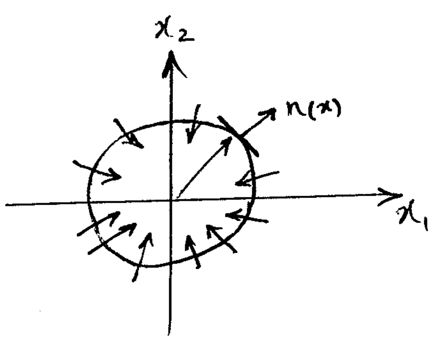
$f_1(x, 0=y) = a\cdot x$

$f_2(x=0, y) = -dy$

Ex 2

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$$

We'll show that, $B_r := \{ x \in \mathbb{R}^2 / \ x_1^2 + x_2^2 \leq r^2 \}$ is positively invariant for sufficiently large r (to be determined)



$$V(x) = x_1^2 + x_2^2 = r^2$$

$$\nabla V(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

$$f(x)\cdot\nabla V(x) = f_1\frac{\partial V}{\partial x_1} + f_2\frac{\partial V}{\partial x_2}$$

$$= 2x_1\left(x_1+x_2 - x_1(x_1^2+x_2^2)\right) + 2x_2\left(-2x_1+x_2-x_2(x_1^2+x_2^2)\right)$$

$$= -2(x_1^2+x_2^2)^2 + 2x_1^2+2x_2^2 - 2x_1x_2$$

$$\leq -2(x_1^2+x_2^2)^2 + 2(x_1^2+x_2^2) + x_1^2+x_2^2$$

$$= -2(x_1^2+x_2^2)^2 + 3(x_1^2+x_2^2)$$

$$= -2r^4 + 3r^2$$

$$= -2r^2\left(r^2 - 3/2\right) \overset{?}{\leq} 0$$

yes if $r^2 \geq 3/2$ (or $r \geq \sqrt{3/2}$)

So $f(x)\cdot\nabla V(x) \leq 0$ if $r^2 \geq 3/2$

Poincare-Bendixon Thm:

Given 2nd order system: $\dot{x} = f(x)$; $x(t)\in\mathbb{R}^2$

M: compact (closed and bounded) set (connected set)

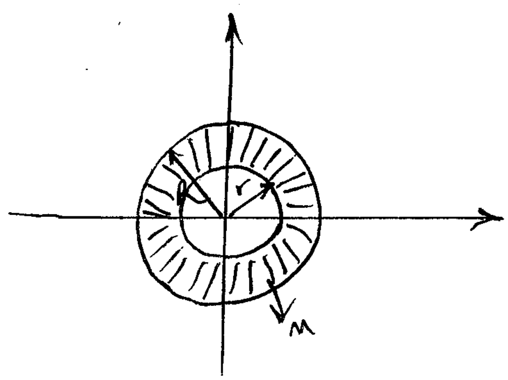
If a) there are no equilibrium points in M, and
b) M is positively invariant

$\Rightarrow$ M contains a periodic orbit.

Ex $\quad \dot{x}_1 = -x_2$

$\dot{x}_2 = x_1$ $\quad ; \quad A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Unique e.p. is the origin $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



$$M = \left\{ x \in \mathbb{R}^2 \,/\, r^2 \leq x_1^2 + x_2^2 \leq R^2 \right\}$$

$$\nabla V(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$f^T(x) \cdot \nabla V = (-x_2 \;\; x_1) \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

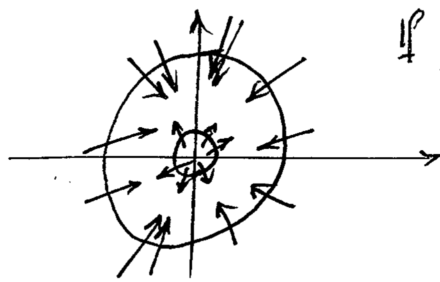
$$= -2x_1x_2 + 2x_1x_2 = 0$$ (on the boundary of M)

M positively invariant ⊕ doesn't contain e.p.

$\Rightarrow$ there is a periodic orbit in M

(in fact, there are ∞ many of them)

Note! It can be shown that the Thm also holds if M contains a single equilibrium point which is either an unstable node or unstable focus.

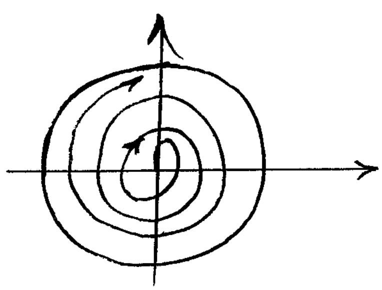


If you take out the e.p.

In example 2 :

$$\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) = f_1$$
$$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) = f_2$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}=\begin{pmatrix}0\\0\end{pmatrix}} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$ unstable focus



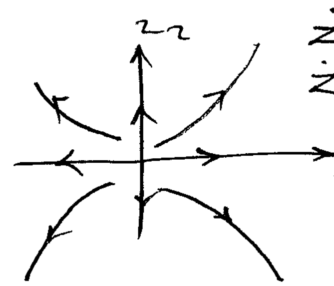
Unstable focus (spiral)

Aside

Unstable node for $n=2$

$\lambda_1, \lambda_2 \in \mathbb{R}$ $\lambda_1 > 0$
$\lambda_2 > 0$

$\dot{z}_1 = z_1$
$\dot{z}_2 = 2z_2$



or