Nonlinear Systems

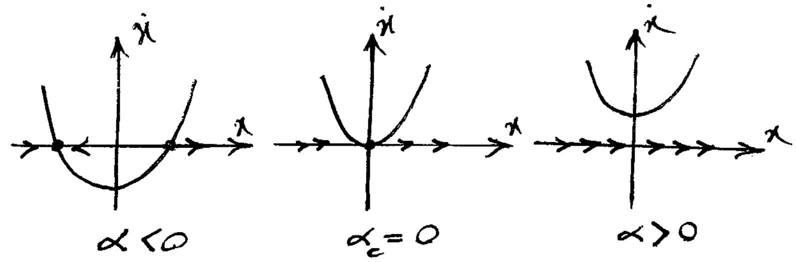
Lecture 03 01,29,13

Last time:

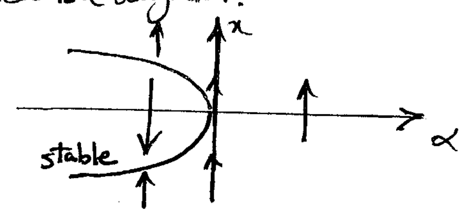
- *Nonlinear phenomena*

1 · Fold bifurcation

$$\dot{x} = \alpha + x^2$$



Bifurcation diagram:



Today:

Transcritical
Pitchfork } bifurcations

Phase portraits of 2nd order systems

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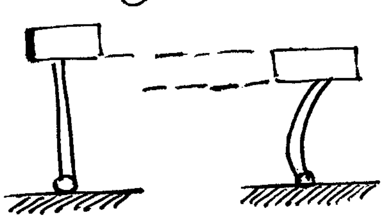
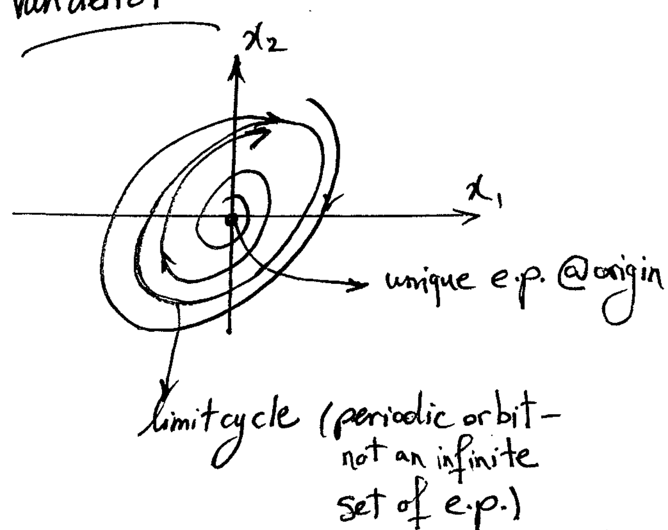
buckling beam:



Vanderpol



② • Transcritical bifurcation

$$\dot{x} = \alpha x - x^2 \qquad x(t) \in \mathbb{R}$$

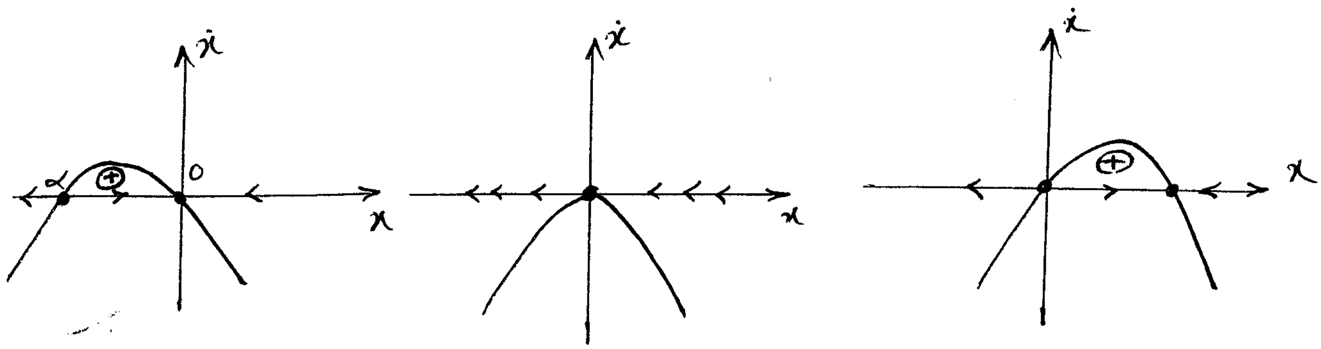
Eq. points:

$$\bar{x}(\alpha - \bar{x}) = 0 \Rightarrow \left.\begin{matrix} \bar{x} = 0 \\ \bar{x} = \alpha \end{matrix}\right\} \text{ 2 eq. points}$$

linearization:

$$\left.\frac{\partial f}{\partial x}\right|_{\bar{x}} = \alpha - 2\bar{x} \quad \begin{matrix} \xrightarrow{\bar{x}=0} \alpha \\ \xrightarrow[\bar{x}=\alpha]{} -\alpha \end{matrix}$$

	$\alpha < 0$	$\alpha > 0$
$\bar{x} = 0$: α	stable	unstable
$\bar{x} = \alpha$: $-\alpha$	unstable	stable

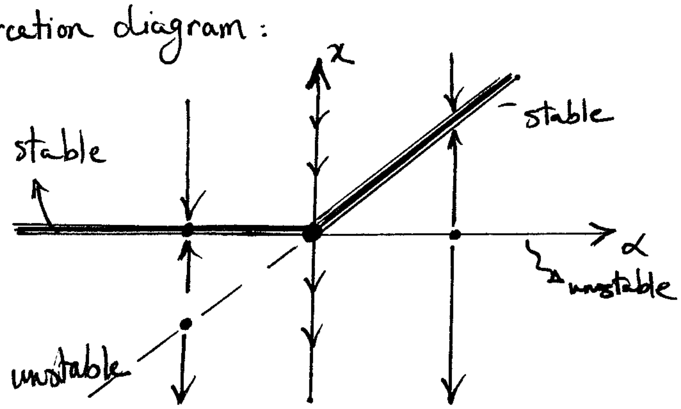


a) $\alpha < 0$ b) $\alpha_c = 0$ c) $\alpha > 0$

origin is stable e.p.

as $\alpha \uparrow$ origin loses stability but another e.p. was formed which was positive

Bifurcation diagram:



Summary: In contrast to fold bifurcation where e.p. dissapears or emerges $\rightarrow (\dot{x} = \alpha - x^2, \alpha\uparrow)$ $\quad (\dot{x} = \alpha + x^2, \alpha\uparrow)$

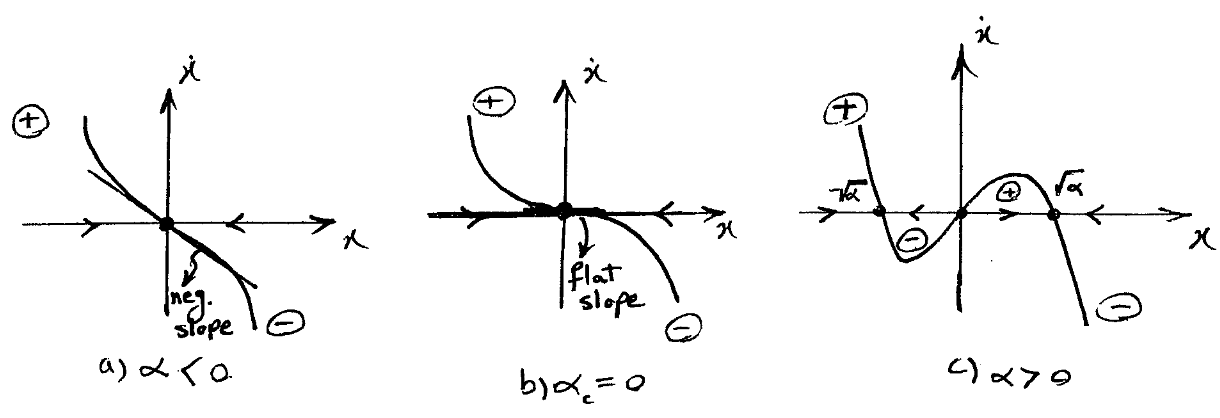
transcritical bifurcation is characterized by change of stability properties

3. Pitchfork

$$\dot{x} = \begin{cases} \alpha x - x^3 & \text{super critical} \\ \alpha x + x^3 & \text{sub critical} \end{cases}$$

3a) $f(x) = x(\alpha - x^2)$

$$\bar{x}(\alpha - \bar{x}^2) = 0 \Rightarrow \bar{x} = \begin{cases} 0 \\ \pm\sqrt{\alpha}, & \alpha > 0 \end{cases}$$

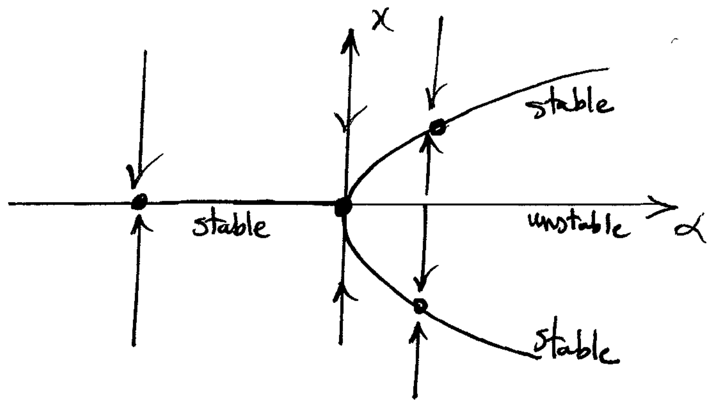


a) $\alpha < 0$ b) $\alpha_c = 0$ c) $\alpha > 0$

* as $\alpha\uparrow$ equilibrium profile lost its stability but no matter where we start we converge to a neighbourhood of the origin.

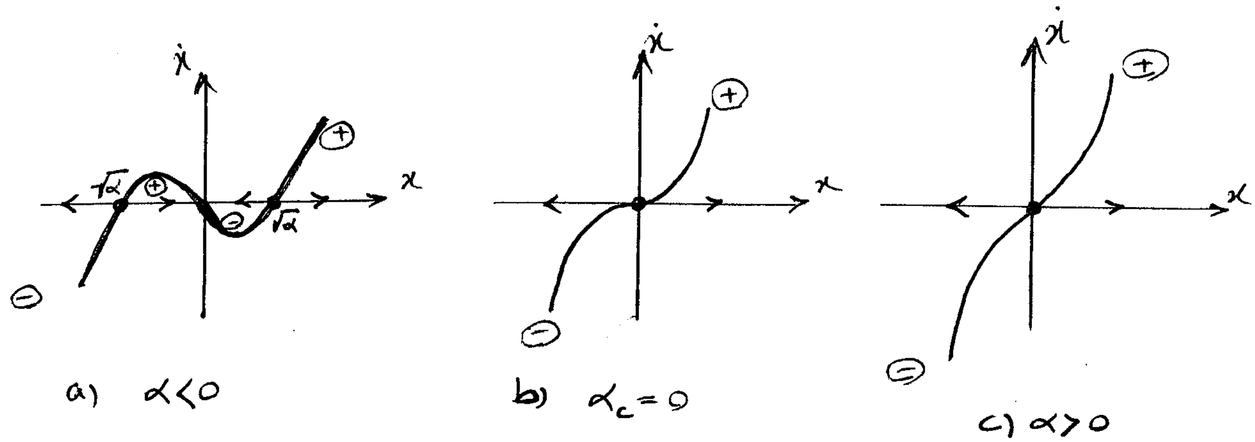
2 equilibrium points emerge when we increase α.

Bifurcation diagram:



3b) $\dot{x} = \alpha x + x^3 = x(\alpha + x^2)$

$$\bar{x} = \begin{cases} 0 \\ \pm\sqrt{\alpha}, & \alpha < 0 \end{cases}$$



* As $\alpha\uparrow$ not only do we lose e.points but we lose stability.
origin is still an e.p. but no matter where you start because of instability of the origin we diverge from the origin.

• 2nd order systems and phase plane analysis

a) Linear systems (unforced)

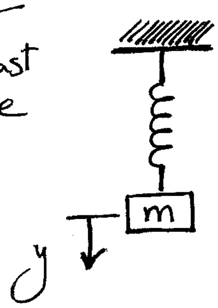
$$\dot{x} = Ax \qquad x(t) \in \mathbb{R}^2$$

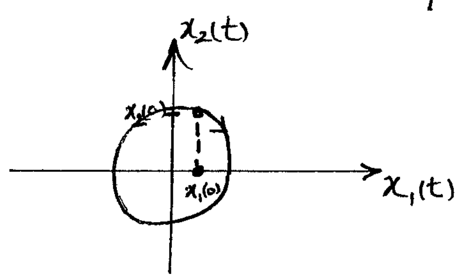
$$\Updownarrow$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\left\{ \begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= -\omega_0^2 x_1 \end{aligned} \right.$$

Ex from last time





$x_2 > 0 \Rightarrow x_1 \uparrow$

Recall : from EE 5231

We can always change coordinates s.t., $\bar{A} = T^{-1}AT$ is brought into the forms below:

a) $\bar{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$; $\begin{matrix} \lambda_1 \neq \lambda_2 \\ \lambda_1, \lambda_2 \in \mathbb{R} \end{matrix}$ (diagonal coord. form)

b) $\bar{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$; $\lambda \in \mathbb{R}$

c) $\bar{A} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$; $\lambda_{1,2} = \alpha \pm j\beta$ $(\beta > 0)$

a) $\dot{z} = \bar{A} \cdot z$

$$\dot{z}_i = \lambda_i z_i \quad ; \quad \lambda_i \neq 0$$

$$z_i(t) = e^{\lambda_i t} z_i(0) \Rightarrow e^{\lambda_i t} = \frac{z_i(t)}{z_i(0)} \Bigg|_{z_i(0) \neq 0}$$

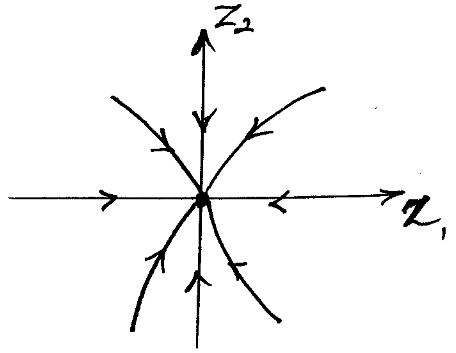
$$z_2(t) = z_2(0) e^{\lambda_2 t} = z_2(0) \left[e^{\lambda_1 t} \right]^{\lambda_2/\lambda_1} = z_2(0) \left[\frac{z_1(t)}{z_1(0)} \right]^{\lambda_2/\lambda_1}$$

$$\Rightarrow z_2 = c \, z_1^{\lambda_2/\lambda_1}$$

$$c = \frac{z_2(0)}{[z_1(0)]^{\lambda_2/\lambda_1}}$$

assume $\lambda_1 < \lambda_2 < 0$ (e.g. $-2 < -1 < 0$)

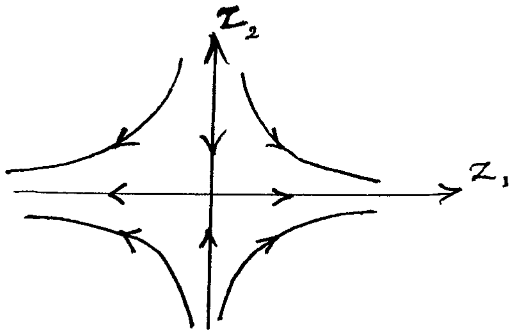
$z_2 = c z_1^{1/2}$



"stable node"

now if $0 < \lambda_2 < \lambda_1$ we will have a similar phase plane portrait but the direction of arrows will change and we will have an "unstable node"

If we have $\lambda_2 < 0 < \lambda_1$ we will have,



"Saddle"