HIGH RESOLUTION SPECTRAL ANALYSIS

Tryphon Georgiou
Electrical & Computer Engineering
University of Minnesota

Spectral Analysis ∼ Analytic Interpolation

Motivation

Theoretical advances

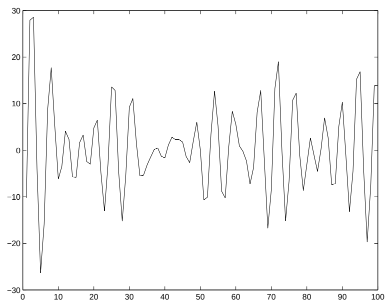
Tools for high resolution analysis

Applications: SAR, ultrasound

- **Signal analysis/filtering**

Given time-series data:

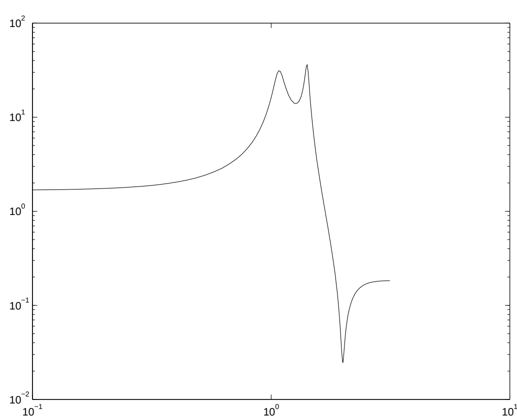
$$\{u_0, u_1, u_2, \ldots, u_{N-1}\}$$

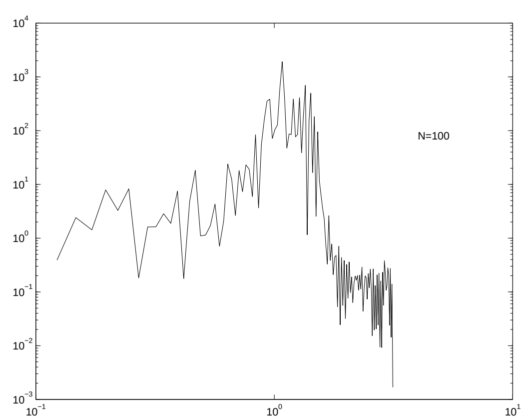


determine the **power spectrum of** y,
i.e., periodicities and "color"

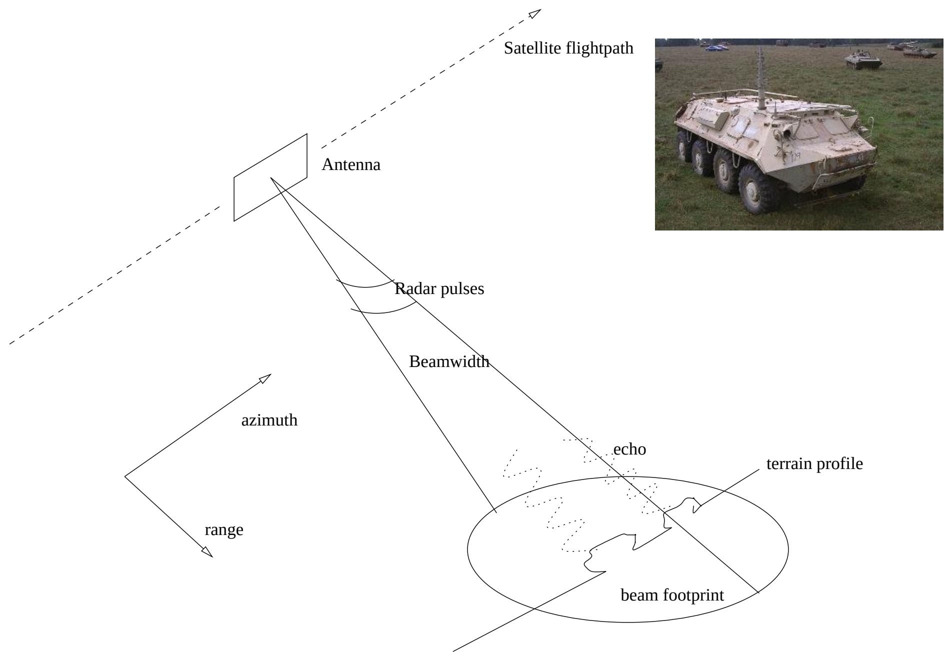
Methods:

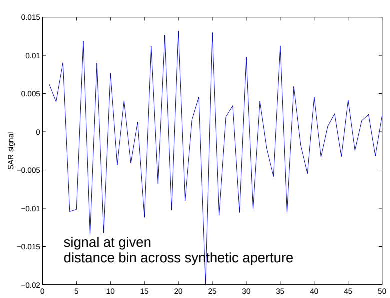
- ◦ Periodogram, FFT
- ◦ Model based (ARMA,....)
- ◦ Modern nonlinear (Maximum-entropy, maximum-likelihood,...)



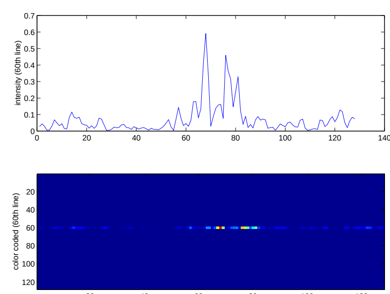


SAR:

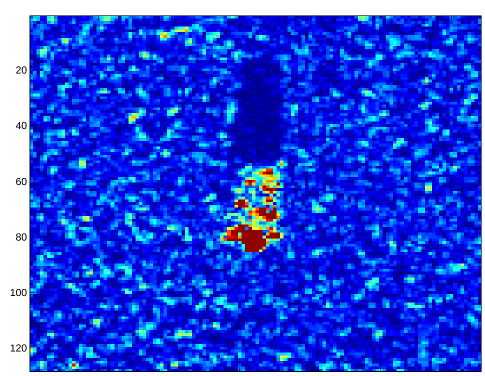




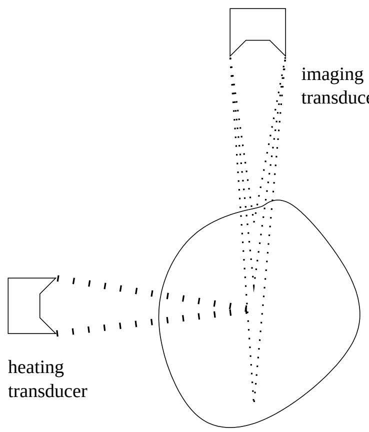
⇒



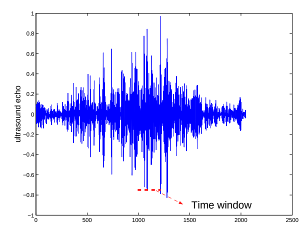
Line by line produces:

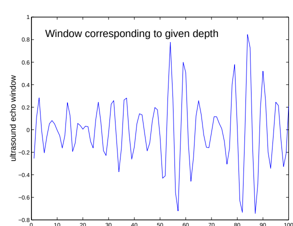


Ultrasound – Noninvasive temperature sensing

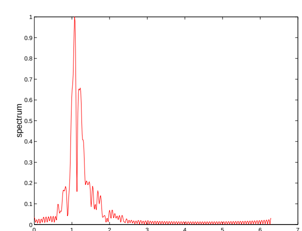


Ultrasound echo:

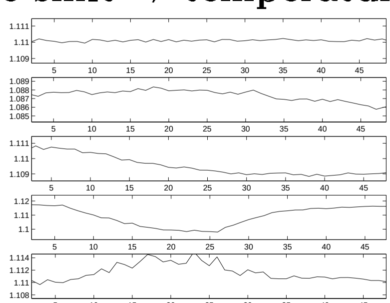




$\Rightarrow$



harmonic shift $\Rightarrow$ temperature profile



- **Reference/collaboration: E. Ebbini**

Moment problem

Find "*mass density*" $\rho(x)$:

$$\int_I x^n \rho(x) dx = c_n, \text{ for } n = 0, 1, \ldots, N.$$

Analytic interpolation

Find f analytic, having $Re(f) > 0$, and

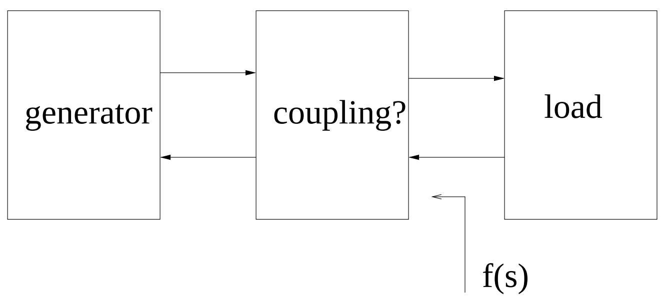
$$f(z) \sim c_0 + c_1 z + c_2 z^2 + \ldots + c_N z^N + \ldots.$$

- $\rho \sim Re(f)$
- Analogous problems for $|f| < 1$, and $f(z_i) = w_i$.

Circuits:	positivity	$\Leftrightarrow$	passivity
Control:	contractiveness	$\Leftrightarrow$	signal attenuation
Signals:	positivity	$\Leftrightarrow$	admissible probability structure

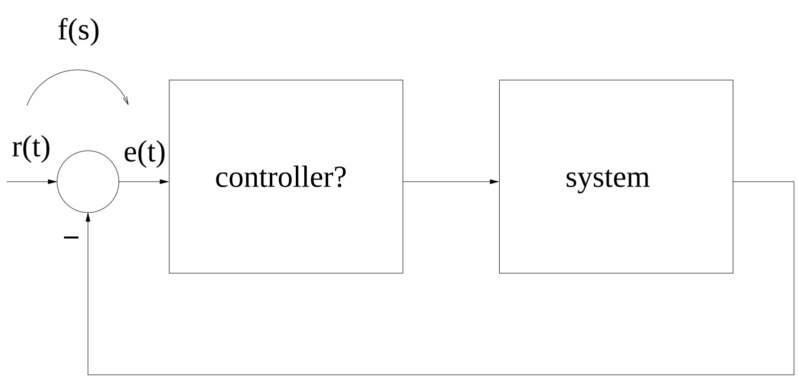
Circuit theory (Fano, Youla-Saito, Helton, . . .)

Maximal power transfer



Control (Zames,Tannenbaum,Doyle,Francis,Helton, . . .)

Robust performance, etc.



Signal analysis (Levinson, Burg, Pisarenko,. . .)

Modeling from covariance statistics,...

$$u_k = \int_{-\pi}^{\pi} e^{jk\theta} dv(\theta)$$

$dv(\theta)$	"amplitude of complex sinusoids"
$\rho(\theta)d\theta \sim E\{dv(\theta)^2\}$	"energy density across frequencies"

Covariance statistics & spectral density

$$c_k = E\{u_t u_{t+k}\}$$

$$\Updownarrow$$

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-jk\theta} \underbrace{\rho(\theta)d\theta}_{d\mu(\theta)}$$

$c_0, c_1, c_2, \ldots$ autocorrelation sequence

$$\Updownarrow$$

$f(z) = \frac{1}{2}c_0 + c_1 z + c_2 z^2 + \ldots$ "positive-real" function

$$\Updownarrow$$

$$\begin{aligned} \rho(\theta) &= \mathrm{Re}(f(e^{j\theta})) \\ &= \ldots c_2 e^{-2j\theta} + c_1 e^{-j\theta} + c_0 + c_1 e^{j\theta} + c_2 e^{2j\theta} + \ldots \geq 0 \end{aligned}$$

RESULTS: **Given finite data** $c_0, \ldots, c_N$:

- PARAMETRIZATION OF SOLUTIONS:

  $\rho = \mathrm{Re}\left(\frac{A+BQ}{C+DQ}\right)$ with Q a "free" parameter

- SPECIFIC CHOICES OF Q LEAD TO DIFFERENT "METHODS":

Maximum Entropy/central solution
maximizing $\int \log(\mathrm{Re}\; f(e^{j\theta}))d\theta$

Pisarenko/MUSIC/ESPRIT
...

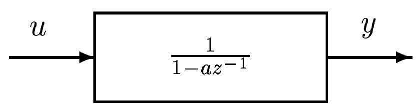
- **Generalized covariance/statistics**
  Collaboration with Chris Byrnes and Anders Lindquist

  Local improvement of resolution

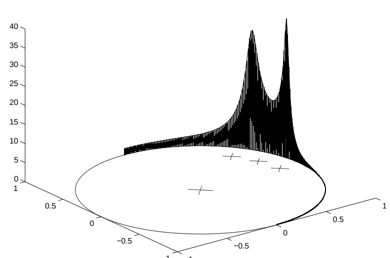
  Spectral analysis $\sim$ problem in generalized interpolation

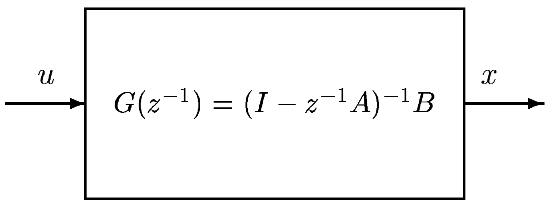
- **Multivariable theory** $\Leftarrow$
- **Special canonical interpolants** $\Leftarrow$
- **Applications** $\Leftarrow$

$$
\begin{aligned}
\mathrm{E}\{y(k)^2\} &= \mathrm{E}\{(u(k) + au(k-1) + a^2u(k-2) + \ldots)^2\} \\
&= c_0(1 + a^2 + a^4 + \ldots) \\
&\quad +2c_1 a(1 + a^2 + a^4 + \ldots) \\
&\qquad +2c_2 a^2(1 + a^2 + a^4 + \ldots) + \ldots \\
&= \frac{2}{1-a^2} F(a)
\end{aligned}
$$



$$F(a) = \frac{1-a^2}{2} E\{y(k)^2\}$$



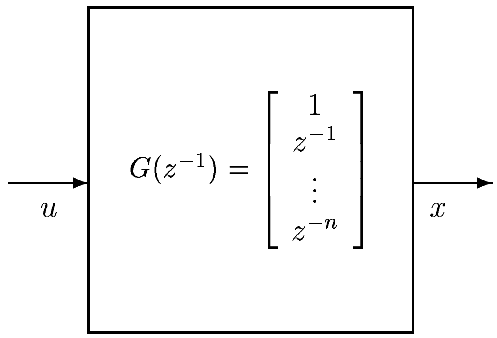


$$x_k = Ax_{k-1} + Bu_k$$

What is the structure of the state-covariance matrix $\Sigma := E\{xx^*\}$?

What are all spectra consistent with Σ?

$$\Sigma \;=\; \int_{-\pi}^{\pi} \left(G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right)$$



- $G(z^{-1})$ "steering vector" of a uniform array

$$x_k = \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ uk - n \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} c_0 & c_1 & \dots & c_n \\ c_1^* & c_0 & \dots & c_{n-1} \\ \vdots & \vdots & & \vdots \\ c_n^* & c_{n-1}^* & \dots & c_0 \end{bmatrix} \text{ is (block) Toeplitz}$$

- "Steering vector" $G(z^{-1}) = (I - z^{-1}A)^{-1}B$ with nontrivial dynamics

$$\begin{aligned}\Sigma &= \int_{-\pi}^{\pi} \left(G(e^{j\theta}) \frac{d\mu(\theta)}{2\pi} G(e^{j\theta})^* \right) \\ &\vdots \\ &= BH + H^*B^* + A\Sigma A^*\end{aligned}$$

With (A, B) controllable pair, A stable, $\Sigma \geq 0$:

Σ **is a covariance of** $x_k = Ax_{k-1} + Bu_k$

$$\Leftrightarrow$$

$\Sigma = BH + H^*B^* + A\Sigma A^*$ **has a solution** H

$$\Leftrightarrow$$

$$\text{rank} \begin{bmatrix} \Sigma - A\Sigma A^* & B \\ B^* & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B \\ B^* & 0 \end{bmatrix}$$

Let Σ state-covariance and H as before:

All input spectra consistent with Σ are

$$d\mu(\theta) \sim \mathrm{Re}\left(F(re^{j\theta})\right) d\theta$$

where

$$F(\lambda) = F_0(\lambda) + Q(\lambda)V(\lambda) \ \textbf{ is positive-real}$$

Data:
$F_0(\lambda) = H(I - \lambda A)^{-1}B$,
and
$V = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ **is inner,**
$\lambda := z^{-1}$, **w.l.o.g.** $AA^* + BB^* = I$

- Σ is the relevant "Pick" matrix
- LFT parametrization of all F's
- for scalar input $\Sigma = W + W^*$ with W **commuting with** A

Define the entropy functional

$$\mathbb{I}(\mu) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\det(\dot{\mu}(\theta))d\theta$$

Then the unique spectrum maximizing $\mathbb{I}$ is

$$d\mu_o(\theta) := \left(\Phi(e^{j\theta})^{-1}\Omega^{-1}\left(\Phi(e^{j\theta})^{-1}\right)^*\right)d\theta$$

where

$$\Phi(\lambda) := (B^*\Sigma^{-1}B)^{-1}B^*\Sigma^{-1}(I-\lambda A)^{-1}B,$$

and

$$\Omega := (B^*\Sigma^{-1}B)^{-1}.$$

- If $\Phi(\lambda) = I - \lambda P_1 - \lambda^2 P_2 + \ldots$

$$\hat{u}_k = P_1 u_{k-1} + P_2 u_{k-2} + \ldots,$$

has minimal variance Ω, in fact it is a **min-max optimal predictor** (i.e., minimizes uniformly over spectra consistent with Σ).

If $u_\ell =$ "white-noise" $+ \sum_{k=1}^{m} \sqrt{\rho_k} e^{j\ell\omega_k}$, then

$$\Sigma = \rho_0 I + \sum_{i=1}^{m} \rho_i G(e^{j\omega_i}) G(e^{j\omega_i})^*$$

Given ANY state-cov. Σ,
$\exists$ a unique decomposition as above
for a minimal value of m

- $\rho_0 = \sigma_{\min}(\Sigma)$, ρ's and ω's obtained via **SVD** of Σ

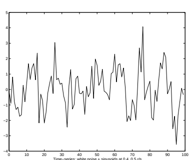
- If $G(z^{-1}) = (1,\ z^{-1},\ \ldots, z^{-n}) \Rightarrow$
Carathèodory-Fèjer/Pisarenko - decomposition
Methods of MUSIC, ESPRIT

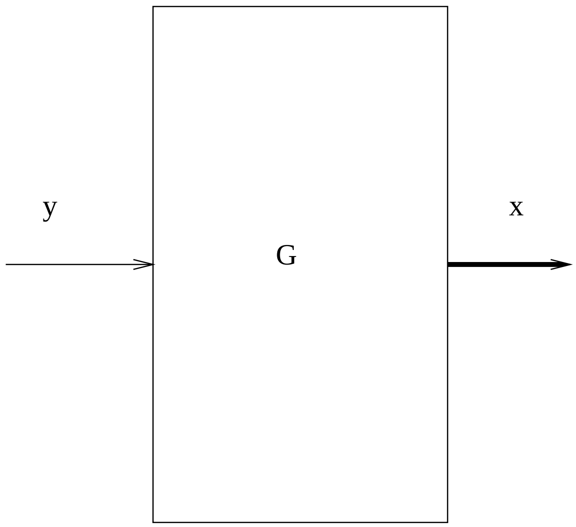
If Σ, Σ_0 are both state covariances for $G(z^{-1})$ and $\Sigma_1 := \Sigma - \Sigma_0 \geq 0$, then Σ_1 is also a state covariance

Pf: $\ldots \Sigma_1 = A\Sigma_1 A^* + BH_1 + H_1^* B^*$ and ≥ 0. $\square$

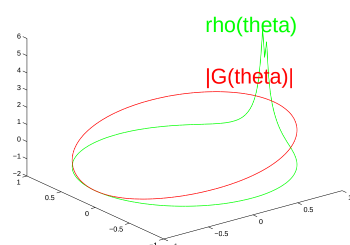
$\Sigma_0 = G(e^{j\theta})G(e^{j\theta})^* \sim$ **complex sinusoidal mass**

$\rho(\theta) = \max\left\{\rho \ : \ \Sigma - \rho G(e^{j\theta})G(e^{j\theta})^* \geq 0\right\} \Rightarrow$ **envelope**



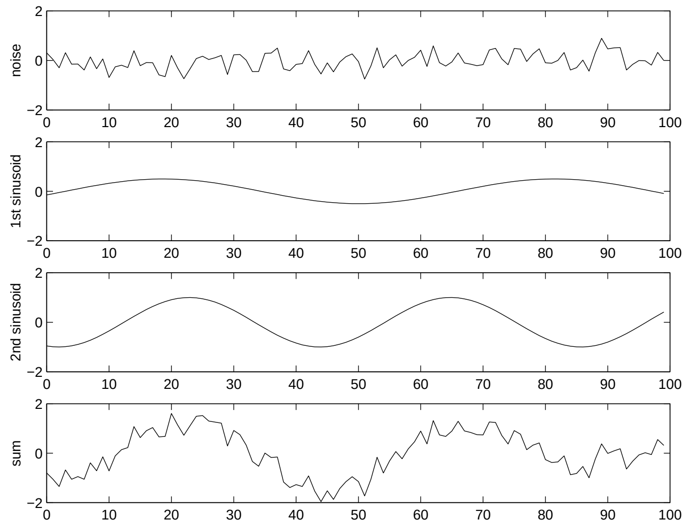


$$\Sigma \sim \sum x_k x_k^*$$



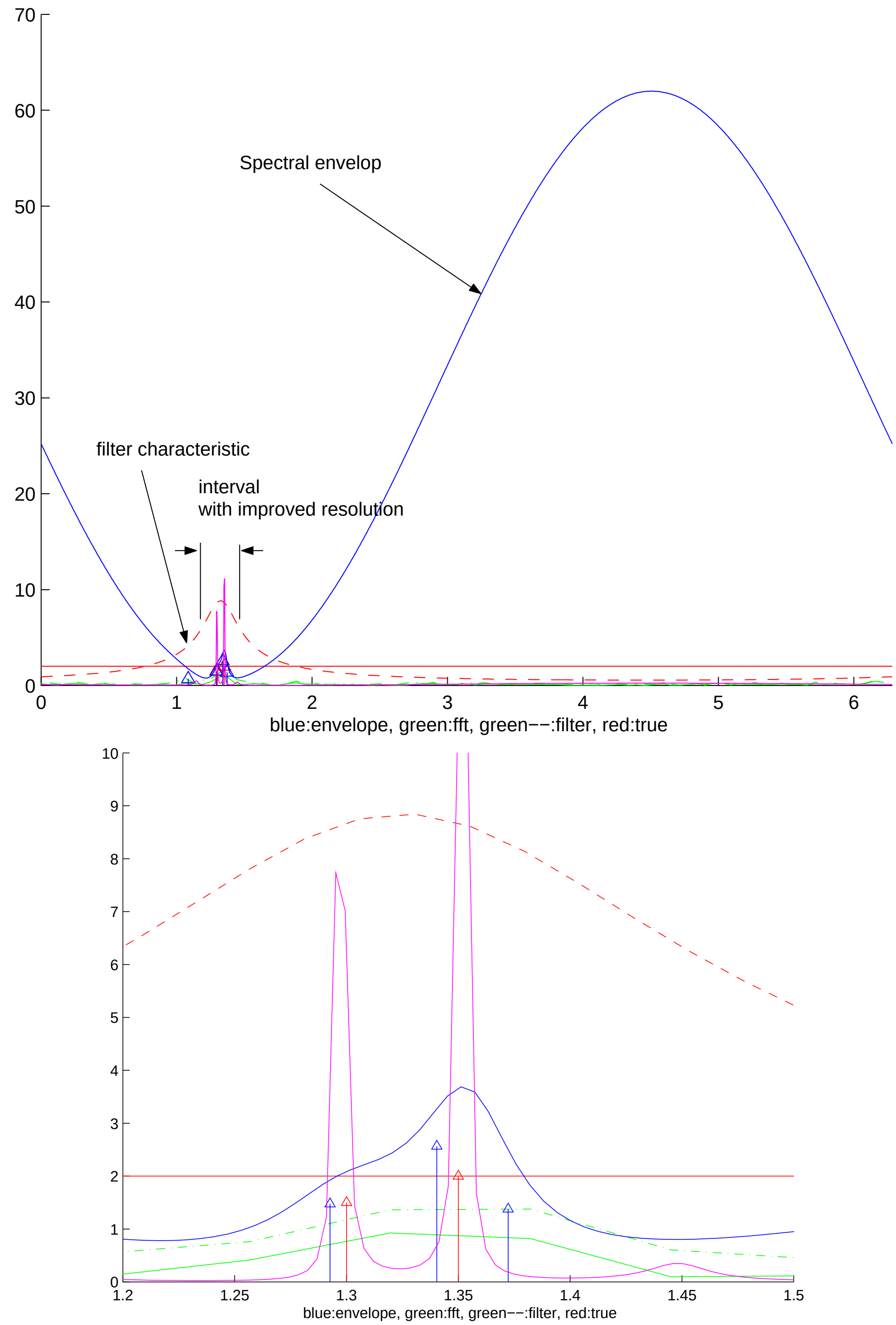
$$\Sigma = \frac{1}{2\pi} \int_{-\pi}^{\pi} (G(e^{j\theta}) d\mu(\theta) G(e^{j\theta})^*)$$

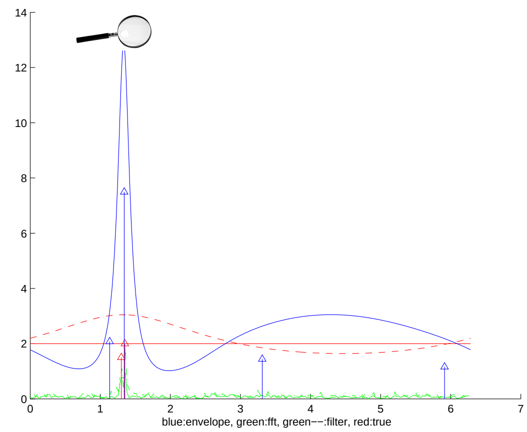
$$\mathbf{u}_k = \nu_k + A_1 \sin(\omega_1 k + \phi_1) + A_2 \sin(\omega_2 k + \phi_2),\ k = 1, \ldots, n,$$



Noise, sinusoid 1, sinusoid 2, and their sum

$$\boxed{\omega_2 - \omega_1 < \frac{2\pi}{n} = \text{Fourier uncertainty bound}}$$


Spectral envelop
filter characteristic
interval
with improved resolution
blue:envelope, green:fft, green−−:filter, red:true
blue:envelope, green:fft, green−−:filter, red:true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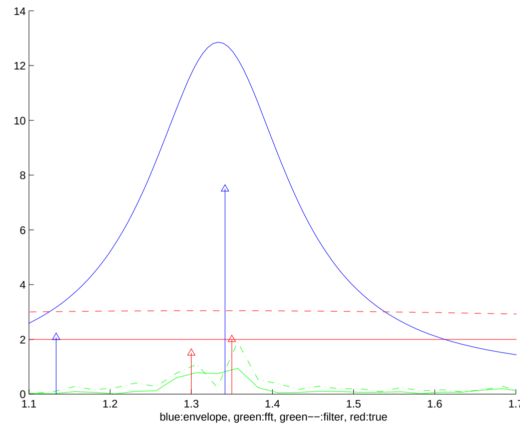
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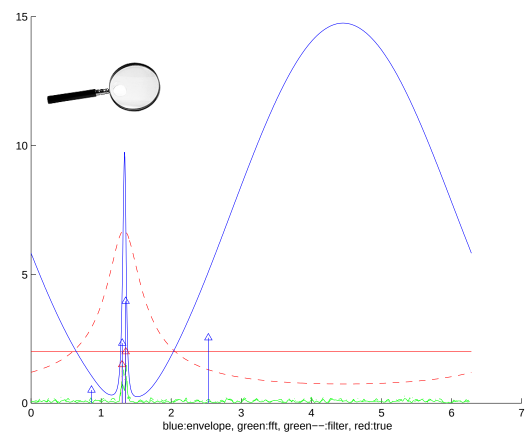
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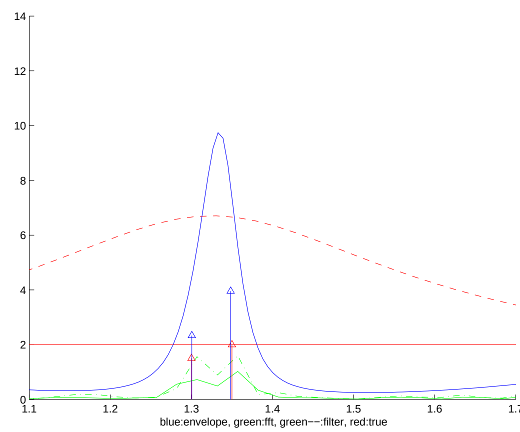
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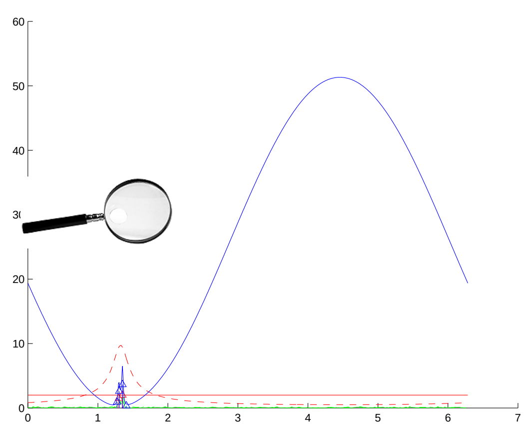
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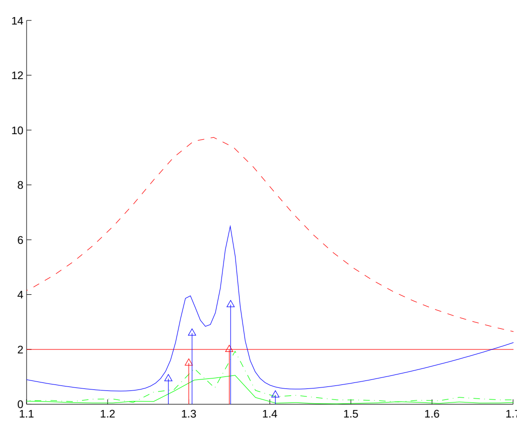
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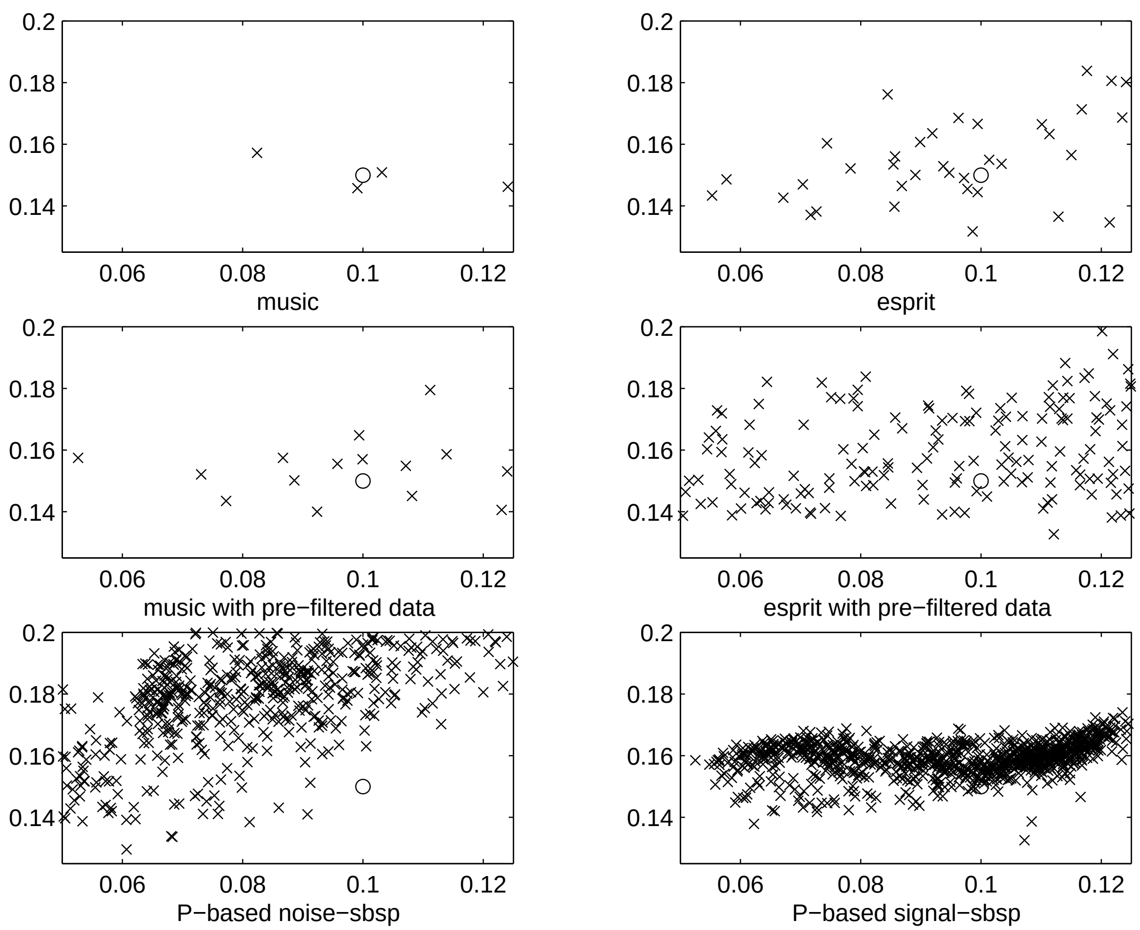
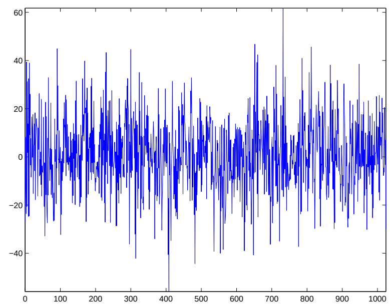
☐ Darts (1000 runs) thrown by:

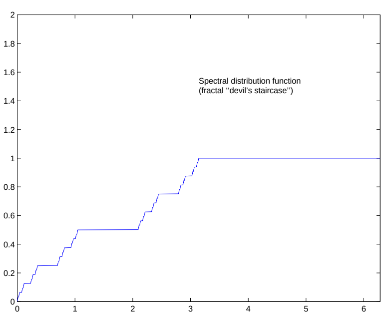
row 1: MUSIC, ESPRIT

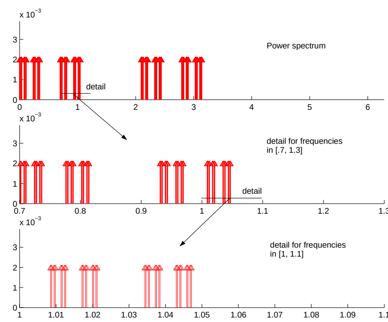
row 2: MUSIC, ESPRIT on "prefiltered data"

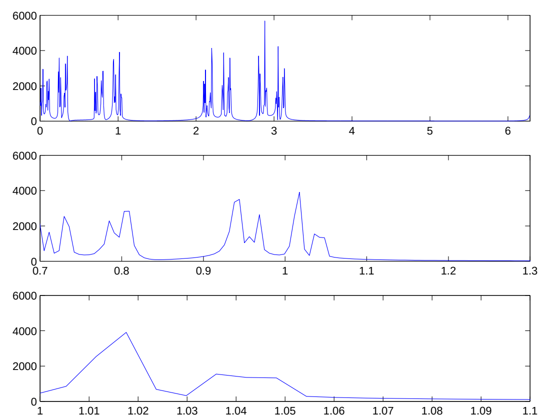
row 3: Boundary interpolant of state-cov data



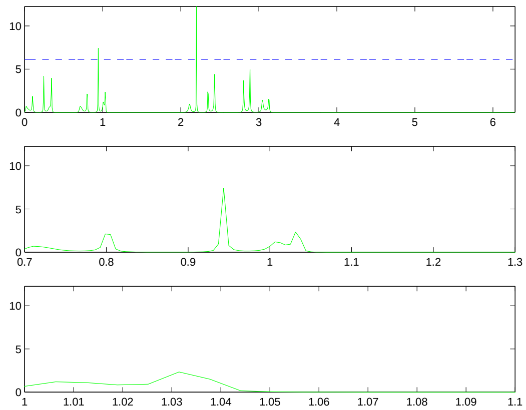


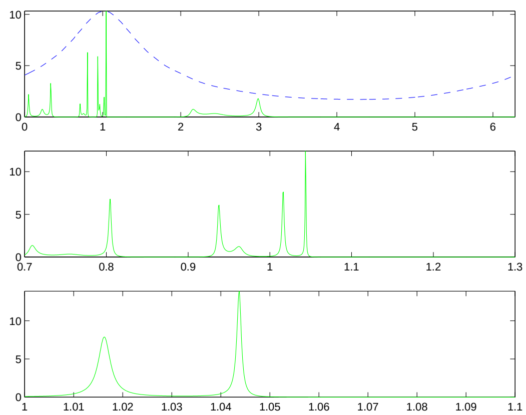






periodogram


10
5
0
0 1 2 3 4 5 6
0.7 0.8 0.9 1 1.1 1.2 1.3
1 1.01 1.02 1.03 1.04 1.05 1.06 1.07 1.08 1.09 1.1


10
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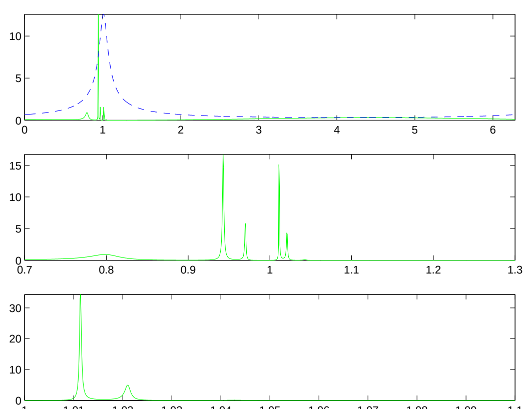
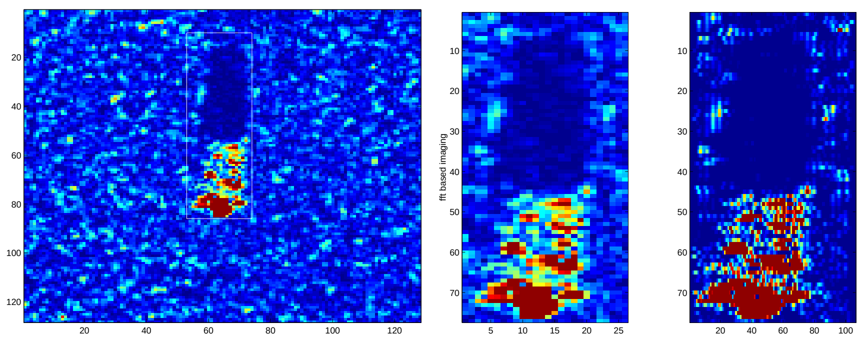
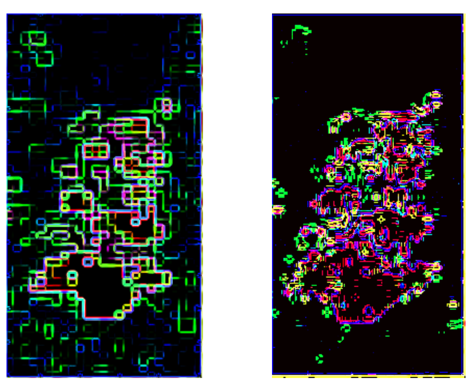
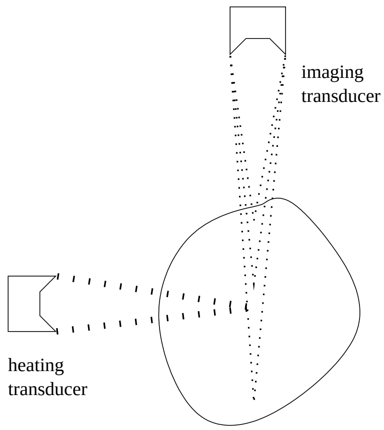

10
5
0
0 1 2 3 4 5 6
15
0.7 0.8 0.9 1 1.1 1.2 1.3
30
20
1 1.01 1.02 1.03 1.04 1.05 1.06 1.07 1.08 1.09 1.1

APC image
MSTAR image of APC
and detail vs. "high resolution" reconstruction

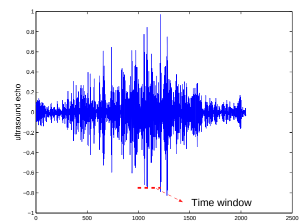


- Using edge detection (MSTAR-image vs. high resolution)

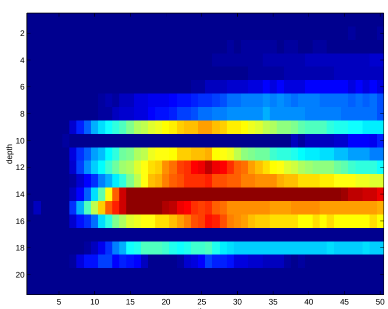


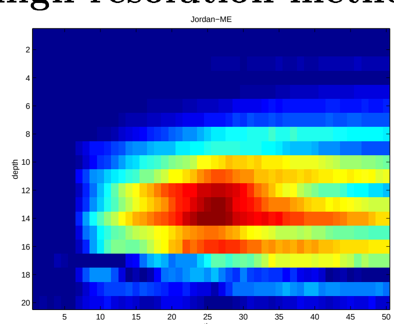


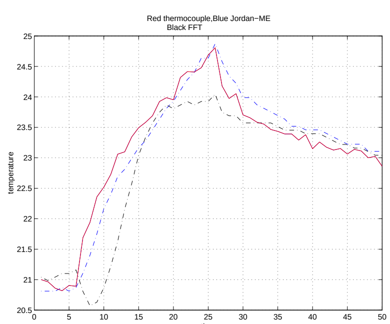
Ultrasound echo:



periodogram analysis vs. high resolution methods







Comparison with thermocouple

(thermocouple-red, periodogram-black, high resolution-blue)

- **Reference/collaboration: E. Ebbini**

Results

- covariance statistics ∼ analytic interpolation
- high resolution methods, applications

Work in progress

- spacio-temporal dynamics and non-uniform arrays
  (Laurent Baratchard-INRIA)
- temperature sensing
  (Emad Ebbini, and A. Nasiri-Amini)
- SAR imaging & target recognition
  (Allen Tannenbaum)
- polarimetric SAR
  (Firooz Sadjadi-Lockheed)

Fundamental questions

- ¿**how can we quantify resolution**?
  ¿fundamental limits beyond Fourier uncertainty?
- tradeoffs between variance and resolution
  seeking an "H_∞-like paradigm"

Matlab code and references at:

`http://www.ece.umn.edu/users/georgiou`