Review of[5]
Classic Papers in Combinatorics
Edited by Ira Gessel and Gian-Carlo Rota
Modern Birkhäuser Classics, 1987 (reprinted in 2009)
492 pages, SOFTCOVER

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1 Introduction

> "We have not begun to understand the relationship between combinatorics and conceptual mathematics." –Jean Dieudonné, A Panorama of Pure Mathematics: As seen by N. Bourbaki, Academic Press, New York, 1982.

The book *Classic Papers in Combinatorics* edited by Gessel and Rota chronicles the development of combinatorics by reprinting 39 papers from the period of 1930 to 1973 – a span of 44 years. These are definitely the formative years for many branches of combinatorial theory. The papers are arranged in a completely chronological order, starting with Ramsey's paper of 1930 (Ramsey theory) and ending with Geissinger's three part paper of 1973 (Möbius functions). The authors of the papers collected in this volume are giants of the field; however this book (rightly) does not try to collect representative papers from all famous combinatorists. A more important goal perhaps is to include representative papers from all areas of combinatorics – and it might have fallen a little short of that goal.

Nonetheless, a bunch of great papers together makes it an excellent reference book. There is a two-page introduction at the start of the book, where the editors try to group the papers according to some common threads – as well as give a brief description of some of the results. This summary reads as if it were a bit hurriedly written, and I could use a longer description of papers and some justification on why these 39 papers clearly stand out among the many excellent papers published in the period. As it stands, this introduction was still quite useful to me for the purpose of browsing through the book.

It is also unclear that, given the book was first printed in 1987, why the editors stop at 1973. Is 1973 the end of the *classic* era in combinatorics by some common agreement? I can surely think of some outstanding results appearing in the seventies and eighties.

The editors also sometime put footnotes in the papers to point out errata or provide some extra relevant information. Such instances are very rare though.

2 Summary

It is clearly not a good strategy to provide summary of all thirty nine papers of the volume here - neither would that be very enlightening. I am providing many representative threads that interest computer scientists in general.

Ramsey's famous paper called "On a problem of formal logic" (Paper 1 of this collection) lays foundation of a large class of existential results that forms Ramsey theory. To quote a later paper by Erdös and Rado, also appearing in this collection (Paper 14), Ramsey's paper "discovered a remarkable extension of this (the very basis pigeon-hole) principle," and can be summarized as follows: "Let S be the set of all positive integers and suppose that all unordered pairs of distinct elements of S are distributed over two classes. Then there exists an infinite subset A of S such that all pairs of elements of A belong to the same class." Although Ramsey theory is very much a topic of textbooks now, it is intriguing to see the ideas developing in the papers by Erdös and Szekeres (Paper 3), Erdös and Rado (Paper 14) and branching out in different directions. The paper "A combinatorial problem in geometry" by Erdös and Szekeres (Paper 3) shows the existence of a number $N(n)$ for any given number n, such that any set containing at least $N(n)$ points has a subset of size n that forms a convex polygon. The graph theoretic result, that is in the core of Ramsey theory, appears in this very paper for the first time: for every large enough graph, there either exists an independent set or a clique of pretty large size.

Erdös's paper "Graph theory and probability," (Paper 19) is perhaps the pioneer of the very powerful probabilistic methods which subsequently motivated developement of many randomized algorithms. This paper starts where Erdös and Szekeres (Paper 3) lefts of. By considering the ensemble average property of all subgraphs of a complete graph, Erdös shows a *converse* result to Ramsey-type theorems: If $n \leq \ell^{1+1/2k}$, then a graph on n vertices exists that does not contain either of an independent set of size ℓ or a k-cycle.

Brooks's "On colouring the nodes of a network" (Paper 7) contains the first nontrivial result in graph-coloring. In a *coloring* of a graph we assign colors to the vertices of a graph such that no two neighbors (connected by an edge) gets the same color. Coloring a graph with a number of colors one more than the maximum degree of the graph is trivial. Brooks shows that a number of color equal to the maximum degree d is actually sufficient provided the graph does not contain a clique with d+1 vertices. The proof is algorithmic. Further results on graph coloring appear in Lovász's "A characterization of perfect graphs" (Paper 34). While it is clear that one needs a number of colors at least equal to the maximum clique size, Lovász shows the conditions under what that limit can be achieved with equality.

There is a number of papers in this volume that are celebrated among the algorithms community – and some results that almost all computer scientists are familiar with. It is nice to see Hall's marriage theorem (Paper 4), Ford and Fulkerson's network flow algorithm (Paper 15) or Edmond's algorithm for matching (Paper 26) in the original papers. I am more used to in seeing Hall's theorem in a graph theoretic formulation. In its original form, the statement involved finding a complete set of distinct representatives for a collection of sets (the graph theoretic statement follows trivially from there). Halmos and Vaughan ("The marriage problem", Paper 11) provides the standard short proof of Hall's theorem.

Ford and Fulkerson's "Maximal flow through a network," (Paper 15) is the easiest read of this collection, despite its far-reaching applications. The maximum flow through a (railway) network is bottlenecked by the minimum cut (a set of links that on removal disconnects the network). The paper first provides a proof of this maxflow-mincut theorem and then turns the theorem into an efficient computational scheme for planar graphs. As an interesting observation, it was shown in the end of this paper that, by constructing a somewhat unusual *dual* graph of any planar graph, the problem of finding a minimum path can be reduced to the maximum flow problem.

Finding most of the natural graph features, such as maximum independent set, minimum vertex cover etc., are intractable computationally and many times even hard to approximate. In contrast, finding a maximum *matching* is very tractable. The algorithm to find maximum matching in polynomial time first appears in Edmond's paper "Paths, trees and flowers" (Paper 26). The philosophical question of why matching in an arbitrary nonbipartite graph tractable bothered Edmond even while writing the paper. The paper contains a rather long digression on what is efficient computability (recall, this is before Karp's or Cook's canonization). At some point Edmond says, "I am claiming, *as a mathematical result*, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph." This is a truly epic paper – perhaps not so much in terms of volume (16 pages), but in terms of beauty and serenity.

This collection does a good work in chronicling the developement of the theory of matroids by including the key papers by Whitney and Tutte. Whitney's paper "Non-separable and planar graphs" (Paper 2) introduces the terminology of *rank, nullity* and *duality* for graphs, and lays foundation for studying linear dependency in combinatorial terms. Matroids are formally defined in the later paper "The abstract properties of linear dependence" (Paper 5) where we see terminologies that are quite standard today. Somewhat nonstandard nomenclature is used in Tutte's "A ring in graph theory" (Paper 9) which redefines some of Whitney's notions in terms of graph theory.

A major topic of this collection is definitely the theory of Möbius functions. In this collection more than a few, including a three-part paper by Geissinger (Papers 37, 38 and 39), deal with Möbius functions. This definitely reflect the editor's interest in the topic. Rota's "Theory of Möbius functions" (Paper 25) is really expository in terms of motivating Möbius functions for the use of enumeration. This set of papers is also the representative of algebraic combinatorics in this collection. I was unaware of Pólya's otherwise famous paper called "On picture writing" (Paper 16). This is definitely one of the most interesting papers and worthy of this collection by any measure. The editors mention in the introduction that this curious paper foreshadows the theory of incidence algebra and Rota's paper (Rota is one of the editors – hence this must be true, although Rota's paper do not cite this paper of Pólya).

Other papers of this collection include Brooks, Smith, Stone and Tutte (Paper 6), that used Kirchhoff's law of electrical circuits to solve a combinatorial problem, Kaplansky's two-page simple solution of the famous *problème des ménags* (Paper 8) and Lubell's short proof of Sperner's lemma (Paper 28).

The few papers that we left out in the above discussion are as important in the development of combinatorics as any one above and represent works of the superstars such as de Bruijn, Dilworth, Katona, Nash-Williams and Stanley, among others.

3 Opinion

As mentioned earlier, all of the papers of this collection are excellent, and this serves as a good reference book. But who would like to have a book such as this? It is unlikely that having this book will, in any way, make life easier for a researcher – as almost all (if not all) of these papers (being very famous) are available over the internet. However a familiarity with the contents of this book might save us the time of internet searching and taking prints of a popular reference several times over the years – we may just find that reference in this collection.

A book like this therefore was somewhat more relevant in 1987 when it first came out. However, even now the *book format* perhaps give some motivation to look back at the original papers without any particular reason. I wish the editors have given some more insight into the papers, some more reasoning to include them, and perhaps share some stories behind them. The complete chronological ordering also makes little sense, as a much better idea would be to group papers that develop a particular topic together – such as papers of Ramsey theory, or papers on matroids. That way the collection would be easier to read and it would be a simpler task to find common themes and techniques out of this collection as a whole.

The editors have done a commendable job of finding a right mix of papers that divides evenly between papers that are problem-solving oriented and papers that focus on theory development. That being said this book may not be a representative of all areas of combinatorics – it is doubtful that if any one book can be. There are plenty of developments in algebraic combinatorics or additive number theory around the time frame considered here. Also the beautiful theories of combinatorial design or finite geometry are absent. Nonetheless, the chosen papers had a huge impact on combinatorics and beyond.

Some papers, such as Erdös and Szekeres (Paper 3) and Hall (Paper 4), are a bit difficult to read because of the poor typesetting. Actually, it seemed to me that every paper is reproduced as their original format and not really reprinted. Therefore there is no consistency in the print sizes of different papers.

Despite of all these, I feel glad that I have this book in my shelf – very few things beat thirty nine of the best papers of last century together under one cover.